Honors Trigonometry

^^^^^^^^^^

Table of Contents

^^^^^^^^^^

Preface . . . . page 2
Table of Trigonometry Identities . . . . page 5

1 Angles . . . . page 10
2 Trigonometry Functions and Inverse Functions . . . . . . page 13
3 The Pythagorean Identity . . . . page 16
4 General Problem set 1 (Trigonometry Basics) . . . . . . page 18

5 Law of Cosines / Law of Sines . . . . page 24
6 Deriving Identities Geometrically . . . . page 33
7 Deriving Identities Analytically . . . . page 36
8 Periodic Events and Functions / Distance Rate & Time . page 55
Trigonometry Graphing Introduction

9 Inverse Trigonometry Functions . . . . page 62
10 Graphing Trigonometry Functions and their Inverses . . page 72
11 Sinusoidal Multiplication and Addition . . . . page 84
12 Polar: Coordinates, Functions, Graphing . . . . page 103
13 de Moivre's Formula and Complex Numbers . . . . page 111
14 Trigonometry Expression Manipulation . . . . page 120
Trigonometry Equation Solving
15 General Problem Set 2 (Advanced) . . . . page 141

16 Dot Product / Cross Product / Navigation . . . . page 150

Index . . . . page 167

Answers to Problems . . . . page 172

10Mar2012,Sat

^^^^^^^

Preface

^^^^^^^

Students should successfully complete a course in proof based; Classical Geometry, Coordinate Geometry and Algebra II before taking this course in honors level Trigonometry. This trigonometry book has its basis in a collection of lessons I prepared to tutor my oldest son Trigonometry. I had also tutored him in Geometry, using the Moise and Downs Geometry text. As a result of the high quality math education he received, in the math portion of his ACT test he scored in the top 1% nationally, otherwise he is a good but not an exceptional student (B average). Within math his highest ACT scores were in the Geometry / Trigonometry category. I mention all this as an endorsement of this book. This book is well able to help any person of reasonable intelligence and dedication gain a thorough understanding of trigonometry and help increase their level of mathematical problem solving ability so they too can score in the top few percent nationally. This book is designed for secondary (Junior High and High School) level students who desire an in-depth honors level math education. These are students who will go on to become our future engineers, scientists, mathematicians and scholars.

Several years ago, students in Utah who took Algebra I and Geometry were those students who wanted to take higher level math, these were not required subjects as they are now. Teachers could teach these classes at higher levels because they were teaching students who were better prepared and more highly motivated. Now teachers must accommodate students who aren't prepared or motivated to take an honors level math course, therefore math isn't taught at the honors level. Algebra I and Geometry classes taught in our Utah schools has become dumbed down compared to how they used to be taught. It is the opinion of this author that the opportunity to take honors level Algebra I and Geometry (proof based) should again be made available to all Utah students.

Teaching Algebra I and Geometry at lower levels has resulted in subsequent math classes, and physics classes being taught at lower levels. It used to be routine for students to prove identities in trigonometry and formulas of motion and kinetic energy in physics, now it is rare. Honors level students shouldn't just be handed formulas in math or physics. They should prove theorems and derive formulas so they know how to do it, and so they have an understanding of how facts in math and physics are discovered and confirmed. Anything short of this is a shallow math and physics education that doesn't help prepare our future scientists and engineers to be world class.

It used to be the norm in secondary math and physics that easy, medium and difficult problems were given, now medium level problems are less common and the challenging problems are gone. My second oldest son tells me when he learned to graph trigonometry functions in school (in 2009), all graphing was done using a graphing calculator. This is unfortunate, because important insights are learned and reinforced graphing (trigonometry) functions by hand and learning and proving the theorems necessary to do this. This book has students do all graphing problems by hand, students then check their work using a graphing calculator or computer.

Students studying this book should attempt to work out all proofs, derivations and problems on their own before looking at the answers. If it is necessary to look at a solution after you have struggled unsuccessfully, look at it only briefly, just long enough to get a needed clue, then continue to work out the problem on your own. The more you do a problem on your own, by trying and perhaps failing, pondering and struggling, the more benefit you will get by doing the problem. Once you have completed a problem, look at the solution in the answer section, in many cases you will get additional perspective on problem solving by looking at other solutions.

Problems in this book are numbered, 1) 2) 3) etc. Problems that have answers available in the answer section will be numbered in the following manner. 1.5) 2.7) 8.13) etc. The number before the decimal point is the problem number. The number after the decimal point is the section number. To locate the solution to a problem in the back of the book, { 3.8) for example } do a search on   3.8) This will take you to the solution. To return to the problem, do a search on 3.8) again. Note: Some problems may be numbered as 3.8.2) (or similar), if so do a search on 3.8.2).

Coordinate Geometry and Trigonometry provide new techniques for solving Classical Geometry type problems, both can be thought of as extensions of Classical Geometry. From the perspective of solving Geometry type problems: Coordinate Geometry and Trigonometry compliment each other. Coordinate Geometry is more versatile and solves a wider range of problems, while Trigonometry is better suited to solve problems involving angles and triangles. Using coordinate geometry and trigonometry alone or together one can solve many if not all classical geometry problems. Several problems in this book are solved using both.

Trigonometry can be divided into two categories. One being an extension of Geometry, the other being those topics in Trigonometry that are not clearly an extension of Geometry. A divide also exists in Coordinate Geometry. Coordinate Geometry addresses problems one expects to find in Classical Geometry. Analytic Geometry deals with several shapes and geometrical objects not found in Classical Geometry. The dividing line between the two is fuzzy. Analytic Geometry deals with circles and parabolas, but also deals with ellipses, hyperbolas and other shapes unique to Analytic Geometry. This book tries to put sections related to geometry towards the front. Sections less related to Geometry are towards the back. Using this ordering of topics, the first parts of this book are close to being a continuation of Geometry and Coordinate Geometry. I recommended the ordering of this book be preserved.

This book of Trigonometry takes an analytic approach, i.e. algebraic techniques are used to develop all identities and solve all problems .. where possible. Most trigonometry books which take a more geometric approach. Analytic proofs as a rule prove all cases of a theorem in one simple proof. Classical geometry proofs have to prove each case separately. No doubt a student should study both approaches. The main development of identities in this book is analytic. In this book one chapter is devoted to deriving identities geometrically. Algebra can be a powerful tool in solving geometry problems. It is interesting to see both approaches, how two very different approaches can solve the same problem. A student studying this book will find their algebra skills becoming significantly strengthened. Algebra skills gained by studying this book help lay an excellent foundation for higher level mathematics and physics.

Text books in America have become dumbed down over the past few decades. (Since about the mid 1980's). Geometry is the most affected, then trigonometry, then algebra. Honors level math text books are not easy to find any more unless you know what to look for. Below is a list of excellent recommended text books.. This author has taught his children from the Singapore math books, (grades 1 through 6) they are decent but in this author's opinion, the American math educational approach used prior to 1980 was superior to Singapore's now. The American approach laid a better arithmetic foundation early on (first things first), and waited to do intensive problem solving until a later age when students minds were more mature.

Arithmetic (grades 1-6) by Upton and Fuller (out of print)
Intermediate Algebra by Louis Leithbold (out of print)
Geometry by Moise and Downs (out of print, but texts are available)
Kiselev's Geometry I and II by Alexander Givental (in print)
Plane Geometry by Shute, Shirk, Porter (out of print)
A Course in Geometry by Weeks and Adkins (in print)
Honors Coordinate Geometry by Curtis Blanco (PDF book available)
Honors Trigonometry by Curtis Blanco (this book)

^^^^^^^^^^^^^^^^^

Table of Trigonometry Identities

^^^^^^^^^^^^^^^^^

Students should become familiar with all these identities. Students will be given the opportunity to derive each of these identities. Identities with stars (i.e. *'s) should be memorized after they have been derived. Notice there is a star by the addition identities, but no star by any of the subtraction identities. Although the subtraction identities are just as important as the addition identities, each of the subtraction identities can be quickly derived form the corresponding addition identity, by keeping in mind that cosine {i.e. cos(o)} is an even function and sine {i.e. sin(o)} is an odd function.

```
* cos^2(o)+sin^2(o)=1 .. Pythagorean Identity  (Pages 16,17)
  1-sin^2(o)=cos^2(o)    These second two forms of the Pythagorean
  1-cos^2(o)=sin^2(o)    identity are seemingly trivial variations
                         of the first but it is important students
                         learn to recognize these forms on sight.
```

```
A^2=B^2+C^2-2BC*cos(a) .. Law of Cosines    (page 24)
```

A,B and C are sides of a triangle and a is the angle opposite side A. The law of cosines is an extension of the Pythagorean theorem. The Pythagorean theorem applies only to right triangles, the Law of Cosines applies to all triangles.

```
sin(a)   sin(b)   sin(c)
------ = ------ = ------ .. Law of Sines    (page 26)
  A        B        C
```

A,B and C are sides of a triangle, a,b and c are the respective angles opposite each of these sides.

```
* cos(-o)= cos(o) .. cosine is an even function  (Page 39)
* sin(-o)=-sin(o) .. sine is an odd function  (Page 40)
  tan(-o)=-tan(o) .. tangent is an odd function
```

```
                                        (Page 40) (Page 38)
* cos(a+b)=cos(a)cos(b)-sin(a)sin(b) .. cosine addition formula
  cos(a-b)=cos(a)cos(b)+sin(a)sin(b) .. cosine subtraction formula
```

```
                                             (Page 41) (Page 42)
* sin(a+b)=sin(a)cos(b)+sin(b)cos(a) .. sine addition formula
  sin(a-b)=sin(a)cos(b)-sin(b)cos(a) .. sine subtraction formula


            tan(a)+tan(b)
* tan(a+b)= -------------- ..  tangent addition formula  (Page 42)
            1-tan(a)tan(b)


            tan(a)-tan(b)
  tan(a-b)= --------------  ..  tangent subtraction formula
            1+tan(a)tan(b)

cos(90°-o)=sin(o) (page 38)    cos(o +/- 180°)=-cos(o)
sin(90°-o)=cos(o) (page 41)    sin(o +/- 180°)=-sin(o)
tan(90°-o)=cot(o)              tan(o +/- 180°)=tan(o)

1+tan^2(o)=sec^2(o)              cos^2(o)-sin^(o)=cos(2o)
1+cot^2(o)=csc^2(o)              cos(o)+sin(o)=sin(2o)/2
```

Double Angle, Squared and Half Angle Formulas

```
cos(2o)=cos^2(o)-sin^2(o) .. cosine double angle formulas (page 44)
       = 2cos^2(o)-1
       = 1-2sin^2(o)

sin(2o)=2sin(o)cos(o)  ..  sine double angle formula

           2*tan(o)
tan(2o) = ----------  .. tangent double angle formula
          1-tan^2(o)

-----

           1+cos(2o)
cos^2(o) = ---------  .. cosine squared formula (page 44)
               2

           1-cos(2o)
sin^2(o) = ---------  .. sine squared formula (page 45)
               2

           1-cos(2o)
tan^2(o) = ---------  ..  tangent squared formula
           1+cos(2o)
```

$$\cos\left(\frac{o}{2}\right) = \pm\sqrt{\frac{1+\cos(o)}{2}}$$ .. cosine half angle formula (page 45)

$$\sin\left(\frac{o}{2}\right) = \pm\sqrt{\frac{1-\cos(o)}{2}}$$ .. sine half angle formula (page 45)

$$\tan\left(\frac{o}{2}\right) = \pm\sqrt{\frac{1-\cos(o)}{1+\cos(o)}}$$ .. tangent half angle formula (page 46)

compression theorem f(x) compressed towards the y axis (page 74) by a factor of 'k' becomes f(kx).

shifting theorem f(x) shifted to the right by an amount of 'a' becomes f(x-a) (page 76)

compression then shifting theorem (page 77)

Sinusoidal Multiplication Identities

$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$$ (page 85)

$$\sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}$$ (page 86)

$$\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}$$ (page 87)

Sinusoidal Addition Identities

$$\cos(a)+\cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$ (page 94)

$$\sin(a)+\sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$ (page 95)

(page 96)

```
                 {    (a+b)       (a+b)}  {    (a-b)       (a-b)}
cos(a)+sin(b)= {cos(---) + sin(---)}  {cos(---) - sin(---)} =
                 {    ( 2 )       ( 2 )}  {    ( 2 )       ( 2 )}
```

```
                    (a+b       )   (a-b       )   (page 97, 11.11.2)
                2cos(--- - 45°)cos(--- + 45°)
                    ( 2        )   ( 2        )
```

(page 100 5.11.3)

Any non shifted cosine function added to any non shifted sine function of the same wavelength or frequency, is a sinusoidal.

```
A*cos(wt)+B*sin(wt)=C*cos{wt-c}
  -- C=sqrt(A^2+B^2)
  -- c is the angle such that cos(c)=A/C and sin(c)=B/C
```

Put an other way, c=invtan(B/A) or c=invtan(B/A)+180°

(page 101)

Any sinusoidal added to anther sinusoidal of the same wavelength or frequency, is a sinusoidal.

```
A*cos(wt-a)+B*cos(wt-b)=C*cos{wt-c}
  -- C=sqrt(CC^2+SS^2)
  -- c is the angle such that cos(c)=CC/C and sin(c)=SS/C
  -- CC=A*cos(a)+B*cos(b) and SS=A*sin(a)+B*sin(b)
```

Put an other way, c=invtan(CC/SS) or c=invtan(CC/SS)+180°

Theorem preliminary to DeMoivre's formula (page 117)

Where A<a and B<b are complex numbers expressed in polar form
magnitude of A<a * B<b = (magnitude of A<a) * (magnitude of B<b)
argument of A<a * B<b = (argument of A<a) + (argument of B<b)

i.e. A<a * B<b = (A*B)<(a+b) . . . {where < means angle}

DeMoivre's Formula (page 118)

Where (r,o) = r<o is a complex number expressed in polar form

(r,o)^n = (r^n,n*o)

Dot Product (page 150)

Where |V| is the magnitude or length of any vector V, the dot product of two vectors A and B is defined as

dot product (A,B) = A.B = |A|*|B|*cos(o)
where o (0°<=o<=180°) is the angle formed by A and B

This being the case, where A=(x1,y1) and B=(x2,y2)

A.B = x1*x2 + y1*y2

Cross Product (page 152)

The cross product of two vectors A and B is an other vector C, where C is perpendicular to both A and B. The magnitude of C is the magnitude of A times the magnitude of B times the sine of the angle o (0°<=o<=180°) defined by A and B. A x B obeys the right hand rule making A x B one vector, not two possible vectors. (See explanation of right hand rule in dot product / cross product section).

Where V1=(a,b,c), V2=(d,e,f)

V1 x V2 = {bf-ec, cd-fa, ea-bd}

or

Where i=(1,0,0); j=(0,1,0); k=(0,0,1)

The cross product can also be expressed as

```
              ( i  j  k )   (Page 154)
V1 x V2 = det ( a  b  c )
              ( d  e  f )
```

(where det means determinate)

Note: In this book the symbol ! means not. For example A != B means A is not equal to B.

Note: In this book the symbol -> often means implies. For example a -> b would be read a implies b.

^^^^^^

1 Angles

^^^^^^

Important: Before doing any problems in this book, read those parts of the Preface which are underlined and highlighted in red.

Angle Classifications (Revolutions and Radians)

There are two main classifications of angle measurement. They are revolutions and radians.

1) Revolutions: {degrees minutes and seconds} and grads
2) Radians

Revolutions

If the vertex of an angle 'a' lies at the center of a circle. The arc length of the portion of the circle that lies in the interior of 'a', divided by the circumference of the circle, is the measure of angle 'a' in revolutions. (A revolution, like a degree, is a unit of angle measurement). For example, if the vertex of a right angle is at the center of a circle, 1/4 of the circle lies in the interior of the right angle, therefore a right angle is 1/4 of a revolution.

By definition a degree is 1/360th of a revolution. A minute is 1/60th of a degree and a second is 1/60th of a minute. In Europe they commonly use grads instead of degrees as a unit of angle measurement. A grad is 1/400th of a revolution.

Radians

The second major classification of angle measurement is the radian. If the vertex of an angle 'a' lies at the center of a circle, the arc length of the portion of the circle that lies in the interior of 'a', divided by the radius of the circle, is the measure of angle 'a' in radians. For example, if the vertex of an angle 'a' lies at the center of a circle, and if the arc length of portion of the circle that lies in the interior of angle 'a' is 1 * the length of the circle's radius, then the measure of angle 'a' is 1 radian.

Pi is a an irrational number that equals approximately 3.142. The definition of Pi is circumference of a circle divided by the diameter of the same circle. Implying Pi = circumference/(2*radius), implying circumference=radius*2Pi. This means the angle measurement of an angle encompassing the entire circumference of a circle (once) is 2Pi radians. (1 revolution = 2Pi radians). Therefore a right angle has a measure of 2*Pi/4 or Pi/2 radians.

Use of radians for angle measurement may seem odd, why radians are used will become more apparent when time is the domain of a cosine or sine function, or when you take Calculus.

Abbreviations: revolutions ~ rev; degrees ~ deg;
radians ~ rads

Angle Classifications Problem Set

1) a) What are the two main types of angle measurement? b) Give a definition of each. c) Degrees and grads are an example of which main type of angle measurement? d) Give a definition of a degree, a minute, a second, and a grad. e) What is the definition of Pi? f) Give an approximation of Pi using 4 significant digits of precision. g) Prove that a right angle = Pi/2 radians.

2) a) One revolution = how many: degrees?, minutes?, seconds?, grads? radians? b) A right angle consists of how many degrees?

3) a) In a unit circle, how "long" is an arc whose angle measure is 1 radian? b) A circle has a diameter of D, how "long" is the circumference? c) A circle has a diameter of D, how "long" is an arc of this circle whose angle measurement is 1 degree?

4) Draw a picture of some circles, on these circles mark an arc whose measure is.. (If, for example the arc is 3 1/2 revolutions, mark an arc of 1/2 revolutions, and state in words that the arc is 3 revolutions more than this)
   a) 3/4 rev; 7/3 rev; 1/6 rev; 20/12 rev
   b) 30 deg; 90 deg; 560 deg; 270 deg; 750 deg
   c) 1 rad; Pi rad; Pi/2 rad; Pi/3 rad; 41/3 rad

5) a) What is the circumference of a circle with a radius of 1?
   b) A circle has a circumference of 1, what is its radius?
   c) A circle has a circumference of C, what is its diameter?

6) a) Convert 3.2 revolutions to degrees; to radians.
   b) Convert 17 degrees to revolutions; to radians.
   c) Convert 2 radians to revolutions; to degrees.

7) An vertex of an angle whose measure is 32°, lies at the center of a circle whose radius is a) 1; b) r. How long is the circumference of the circle that lies within this angle?

Angles in Trigonometry vs Angles in Geometry

Consider a ray on the 2 dimensional Cartesian plane with its end point at the origin. Let this ray have a home position of being aligned with the positive x axis. If this ray is rotated from its home position by any angle o (-infinity < o < infinity), the ray is considered to have an orientation of an angle of o. If the ray is rotated in a counter clockwise direction its angle of orientation is considered to be positive. If the ray is rotated in a clockwise direction its angle of orientation is considered to be negative. Trigonometry angles, unlike geometry angles* are not restricted to being greater than 0 degrees, and less than 180 degrees, an angle can have any real numbered value. Using this new definition of angle the following angles exist and are geometrically indistinguishable.

1 revolution vs 2 revolutions vs -1 revolution etc.
0° & 360° & -360° & 720°; 180° & -180°; -1° & 359°
0Pi rad & 2Pi rad & 8Pi rad & -4Pi rad
1/4 Pi rad & 2 1/4 Pi rad & -1 3/4 Pi rad

* The Moise and Down's Geometry book states that angles have a measure of from greater than 0 degrees to less than 180 degrees. What might otherwise be acknowledged as an angle isn't recognized as an angle by the Moise and Downs Geometry book if its measure is outside of these bounds. Other Geometry books might have a different definition of an angle, recognizing angles whose measure is from greater than 0° but less than 360°. Geometry books do not as a rule do not acknowledge angles whose measure is 0° or less, or 360° or more.

Angles in Trigonometry vs Angles in Geometry Problem Set

1) How do angles in geometry and angles in trigonometry differ?

2) State which of the following angles are exclusively trigonometry angles. a) 12°; b) -60°; c) 179°; d) 180°; e; 700°.

3) On a Cartesian coordinate system, draw rays whose endpoint is the origin and whose orientation is the same as each of the angles following angles. a) 1/3 rev; b) -1/10 rev; c) 0°; d) 90°; e) -90°; f) 180°; g) -180°; h) -135°; i) 270°; j) -120°; k) 30°; l) -30°; m) -7500°; n) 126,000°; o) Pi/3 rads; p) Pi/4 rads; q) -7Pi/6 rads; r)100,000 rads s) -7Pi/6 rads; t) -13Pi/6 rads; u) -250,000 rads.

4) Which of the following angle pairs are geometrically indistinguishable. a) 0° and 360°; b) 15° and 345°; c) 20° and -340°.

^^^^^^^^^^^^^^^

2 Trigonometry Functions and Inverse Functions

^^^^^^^^^^^^^^^

'o' is pronounced theta
cos(o) is pronounced cosine of theta.
sin(o) is pronounced sine of theta.
tan(o) is pronounced tangent of theta.

Triangle Definition of Trigonometry Functions cos, sin and tan

Given a right triangle where 'o' an angle opposite one of the legs, cos(o) is defined as (the length of) the leg adjacent angle o divided by (the length of) the hypotenuse. sin(o) is defined as the leg opposite angle o divided by the hypotenuse. tan(o) is defined as sin(o)/cos(o).

I) Without looking, give the triangle definition of a) cos(o); b) sin(o); c) tan(o).

II) In a right triangle, 'o' is an angle opposite one of the legs. Prove: tan(o), is equal to the side opposite 'o' divided by the side adjacent 'o'.

III) A triangle has sides of length 3,4 and 5. Angle 'a' is opposite the side of length 3. Angle 'b' is opposite the side of length 4. What is a) cos(a); b) sin(a) c; tan(a); d) cos(b); e) sin(b); f) tan(b).

IV.2) Use Classical Geometry to prove the 30°-60°-90° triangle theorem.

Circle Definition of Trigonometry Functions cos, sin and tan

Let R be a ray in the Cartesian plane, whose end point is at the origin and whose angle of orientation is 'o'. Let C be the unit circle centered at the origin. (A unit circle is a circle whose radius = 1). cos(o) is defined as the x coordinate of the point where R intersects C. sin(o) is defined as the y coordinate of the point where R intersects C. The circle definition tan(o) tan(o)=sin(o)/cos(o).

I) A ray with its end point at the origin has an orientation of o, what are the coordinates of the point where this ray intersects a) the unit circle centered at the origin? b) the circle of radius R centered at the origin?

II) Without looking give the circle definition of a) cos(o); b) sin(o); c) tan(o).

III) Prove that the triangle definition of cos(o) and sin(o) is consistent with the circle definition of cos(o) and sin(o), i.e. prove cos(o) (triangle definition) = cos(o) (circle definition) (0°<o<90°) and sin(o) (triangle definition) = sin(o) (circle definition) for (0°<o<90°)

IV) Show that a ray with an orientation of o, has a slope of tan(o).

V) Without using a calculator, determine each of the following.
a) cos(30°); b) sin(-30°); c) tan(60°); d) tan(-330°);
e) cos(-630°); f) sin(1050°); g) tan(300°); tan(840°)
h) sin(-1170°); i) cos(-840°).

VI) A ray with an orientation of -15° is geometrically indistinguishable from a ray with an orientation of z degrees. a) Give 2 concrete instances of z (z!=-15°). b) List all possible angles of z degrees. Hint: A person asked to list all positive integers could write 1,2,3 ... . Note: Remember that != means not equal.

Note: The terms cos, sin and tan are abbreviations for cosine, sine and tangent respectively.

Note: It is preferred mathematical notation that cos(o)^2 be written as cos^2(o), likewise for sin and tan.

<u>Table of Trigonometric Functions</u>

cosine function
cos(o) .. triangle definition (0°<o<90°) (previously defined)
cos(o) .. circle definition (-inf<o<inf) (previously defined)

----

sine function
sin(o) .. triangle definition (0°<o<90°) (previously defined)
sin(o) .. circle definition (-inf<o<inf) (previously defined)

----

tangent function
tan(o) = sin(o)/cos(o) (-inf<o<inf) (previously defined)

The following are lesser used trigonometry functions.

secant function   .. sec(o) = 1/cos(o) (-inf<o<inf)
cosecant function   .. csc(o) = 1/sin(o) (-inf<o<inf)
cotangent function .. cot(o) = 1/tan(o) (-inf<o<inf)

It is preferred mathematical notation that sec(o)^2 be written as sec^2(o), likewise for csc and cot.

Exercise: Without looking, list and then provide all definitions of of the 6 trig functions.

Making use of a Calculator to calculate trigonometry functions and inverse trigonometry functions

Trigonometry Functions

A calculator may be used to determine the cosine, sine and tangent of angles. To do this enter the angle into the calculator, then press the cos, sin or tan key. Before doing this it is important to ensure the calculator is in the angle mode you desire, i.e. degrees radians or grads.

For example, suppose you wish to find sin(72°). First ensure the calculator is in degrees mode. If not put it in degrees mode. (Refer to calculator instructions to see how to do this). Next enter the number 72 into the calculator and press the sin key. The calculator will return sin(72°) or 0.95105 ... .

1) Determine the numerical value of each of the following.
   a) cos(-17°); b) sin(114°); c) tan(87°); d) csc(28°); e) cot(58°)
   f) cos(1rad); g) sin(12rad); h) tan(-5/3 Pi rad); i) sec(Pi/3 rad)

2.2) A triangle has a hypotenuse of length 5. One of its angles is 23°. What are the lengths of the legs of this triangle?

Inverse Trigonometry Functions

The following discussion uses tangent, a similar discussion would also apply to cosine and sine and the other trigonometry functions. If an angle has a tangent of 7 {tan(o)=7} and we wish to find the measure of an angle that has this tangent. We first put 7 into a calculator and then press the then press the invtan (or arctan) key. The calculator will then return an angle whose tangent is 7. There exists more than one such angle but the calculator will return only one. The number the calculator returns is the inverse tangent of 7, or invtan(7) or arctan(7).

3) Find all angles o (0° <= o < 360°) such that a) cos(o)= 0.75;
   b) sin(o) is 0.5; c) tan(o)= 7; d) sec(o)=3; e) csc(o)=5.

The Inverse trigonometry functions are invcos(x), invsin(x), invtan(x), invsec(x), invcsc(x), and invcot(x), When seeing any one of these functions, invcos for example. Think the words, the angle returned by a calculator, whose cosine is x, i.e. cos{invcos(x)}=x.

^^^^^^^^^^^^^^^^^

3 The Pythagorean Identity

^^^^^^^^^^^^^^^^^

The Pythagorean identity, Identity I (I1) has 3 important forms, though their differences seem trivial. It is important for students to learn to recognize the Pythagorean identity in each of these three forms.

cos^2(o)+sin^2(o)=1; 1-sin^2(o)=cos^2(o); 1-cos^2(o)=sin^2(o)

Two proofs of the Pythagorean identity are given here. The first is based on the triangle definition of sin(o) and cos(o), the second is based the circle definition of sin(o) and cos(o).

Proof 1 (Pythagorean Identity Triangle Proof)
Let A,B,C represent the sides of a right triangle, and a,b,c represent the angles opposite each of these sides with a being opposite A, b being opposite B and c being opposite C. Let C be the hypotenuse of this triangle. By the Pythagorean theorem we have A^2+B^2=C^2. We divide both sides of this equation by C^2 which gives us

eq 1)

```
[A]2   [B]2  [C]2
[-]  + [-] = [-]
[C]    [C]   [C]
```

From the triangle definition of cos and sin we have

eq 2) cos(a)= B/C and eq3) sin(a)= A/C

substituting eq 2 and eq 3 into eq 1 and simplifying we have

sin^2(a)+cos^2(a)=1 (0°<a<90°)

'a' can represent either of the acute angles of a right triangle therefore in general we have

```
cos^2(o)+sin^2(o)=1  (0°<o<90°) <--- I1 (Identity 1)
                                        Pythagorean identity
                                        (Triangle Version)
```

A ray with its endpoint at the origin, whose orientation is o, {-inf < o < inf} intersects the unit circle centered at the origin at {cos(o),sin(o)} (by definition).

Proof 2 (Pythagorean Identity Circle Proof)

From the circle definitions of cos(o) and sin(o) we have, {cos(o),sin(o)} is a parametric form of the unit circle centered at the origin. The Cartesian equation of the unit circle centered at the origin is x^2+y^2=1, implying that cos^2(o)+sin^2(o)=1.

```
cos^2(o)+sin^2(o)=1  (-inf<o<inf)   <--- I1 (Identity 1)
                                         Pythagorean identity
                                         (Circle Version)
```

<u>Pythagorean Identity Get Acquainted Problems</u>

Make use of the Pythagorean identity to do the following problems.

1) Write 3 versions of the Pythagorean identity from memory.
2) If sin(o)=2/3, and cos(o) is positive, what is cos(o)?
3) If cos(x)=2/5, and sin(o) is negative, what is sin(x)?
4) Express cos(o) in terms of sin(o).
5) Express sin(o) in terms of cos(o).
6) What is the definition of tan(o)? Express tan(o) in terms of
 a) sin(o); b) cos(o);
7) If sin(o)=3/7, and tan(o) is positive, what is tan(o)?
8) If cos(o)=1/9, and tan(o) is positive, what is tan(o)?
9) a) If tan(o)=6, and cos(o) is positive, a) what is cos(o)?;
 b) what is sin(o)?; b) Do problem 'a' again geometrically, i.e. make use of a right triangle figure and the Pythagorean theorem.

10) Prove that {1-cos^2(o)}+{1-sin^2(o)}=1
11) Prove that {1-cos^2(o)}/sin(o)=sin(o)
12) Prove that {1-cos^2(o)}/{1-sin^2(o)}=tan^2(o)

```
               cos(b)-sin^2(a)cos(b)
13) Prove that --------------------- = cos(b)cos(a)
                       cos(a)
```

^^^^^^^^^^^^^^^^^^^^^^^^

4 General Problem set 1 - Trigonometry Basics

^^^^^^^^^^^^^^^^^^^^^^^^

Note: An angle of 45 degrees, 31 minutes, 18 seconds can be expressed as 45° 31' 1''.

1.4) A degree is 1/360th of a revolution. A minute is 1/60th of a degree. A second is 1/60th of a minute. a) Express the angle 17° 42' 18'' as an angle of degrees. (Retain at least 5 significant digits after the decimal point). b) Express the angle 76.43120°, in degrees, minutes, seconds format.

2) a) Convert 12 days, 3 hours, 19 minutes, 41 seconds to days.
   b) Convert 21.7123890 days to days, hours, minutes and seconds.

3) a) Convert 65.21234° to degrees, minutes, seconds; and then do the calculations to convert this answer back to degrees. (Did you get what you started with)? b) Convert 51 days, 13 hours, 16 minutes, 21 seconds to days and back again.

4) Calculate the following
   a) 13°,21',18'' + 31°,8',5''
   b) 56°,45',11'' + 111°,31',58''
   c) 3°,21',3'' - 121°,58',14''
   d) 11days, 14hrs, 17min, 28sec - 15days, 44hrs, 52min, 59sec

5) a) On a unit circle, mark and label the following angles; o= 0°, 30°, 45°, 60° and 90°; b) Now on the same circle, mark and label all angles 'a' where cos(a)= +/- cos(o); c) On this same circle, mark and label all angles 'b' where sin(b)= +/- sin(a).

6) a) For each of the angles listed below, find an angle that is geometrically indistinguishable, between 0 and 1 revolutions. Express this angle in the same form (revolutions, degrees or radians) as the given angle. b) It is possible to check each of your answers in part "a" by making use of the sin and cos functions on a calculator. Make use of a calculator to check these answers. c) C is a circle centered at the origin. R is a ray whose end point is also at the origin. For each of the angles listed below, assume R is oriented at this angle, then mark the location on C, where C and R intersect.

   Revolutions: 11; -21.2; -1057.73
   Degrees: 1002; 10020; -796,629
   Radians: 10; -1; -48001

7) Assume the circumference of the earth is 24,900 miles. a) Two people are on the equator of the earth. One is 63 degrees, 16 minutes, 18 seconds (63° 16' 18'') east of the meridian that passes through Greenwich England. The other is 13.5287 degrees east of this meridian. How far, in miles and feet, would one person need to travel to get to the other person? b) Do the same problem with one person being 1.3214 radians east of this meridian, the and other 0.3798 radians west of it.

8) The radius of a compact disk is 6 cm. Assuming the following rotational speeds, find the linear speed of the edge of the CD. a) 500 revolutions/second; b) 3066 radians/second; c) 169,200° per second.

9) a) A circle of radius 5 centered at the origin circumscribes a regular pentagon. One of the vertices of the pentagon has coordinates (0,5). What are the coordinates of each of the other vertices? b) What is the length of each of the pentagon's sides?

10.4) An arrow is shot at an angle of 37° from the horizontal, at a speed of 175 ft/sec. At the instant it is shot, a) What its horizontal speed? b) What is its vertical speed?

11.4) A triangle has a hypotenuse of length C. The angle opposite one of its legs is o. How long are the legs of this triangle?

12) [If a pool ball's heading is the angle o, then the pool ball is moving, and its movement is in the same direction is as if it had started at the endpoint of a ray whose orientation is o, and from there it is moving along the ray. If a barrier's orientation is the angle o, then the barrier is parallel or collinear to all rays whose orientation is the angle o]. Pool balls move in a straight line until they hit a barrier at the edge of the pool table. After hitting the barrier, they move in a straight line as they retreat from the barrier. The angle that the line of the pool ball's retreat makes with respect to the barrier, is equal to the angle that the line of the pool ball's approach makes with respect to the barrier. a) A pool's ball heading is 58°, it then hits a barrier whose orientation is 13°. What is the pool ball's new heading? b) A pool ball hits a barrier whose orientation is 132° and then retreats along a heading of 35°, what was the heading of the pool ball when it was approaching the barrier? c) A pool ball's heading is 98°, it hits an unseen barrier and then retreats along a heading of -12°, what is the orientation of the barrier?

13) Wanting to determine the height of a flag pole, you position yourself 176 feet from the base of the flag pole. Using an optical instrument you then determine the angle (from the horizontal) from the optical instrument to the top of the flag pole is 21°. The optical instrument is 5 feet above the ground. How high is the flagpole?

14) A helium filled balloon is tied to a 32 meter tether attached to the ground, because of wind the tether makes a 73° angle with ground, how high is the balloon?

15) The hypotenuse and a leg of a triangle form the angle o. The length of this leg is A. How long is the hypotenuse and other leg?

16) A man wishes to determine the width of a river. He marks a point B at the rivers edge which is directly across from a rock (point A) on the other side river of the river and also at the river's edge. 72 feet away from B he marks an other point C such that angle ABC is measured to be 90°. Angle ACB is 53°. How wide is the river?

17) An observer in a lighthouse 350 feet above sea level observes two ships due east. From the horizontal, the angles of depression to the ships are 4° and 6.5°. a) How far apart are the ships? b) How far is it from the observer to each of the ships?

18) An airplane is flying 560 mph horizontally and at a constant heading (always in the same direction). It is spotted by an observer at an angle of 31° from the horizontal. One minute later it is directly overhead. a) How high is the airplane? b) How far was the airplane from the observer when it was first noticed?

19.4) Two ranger stations a and b are 38km apart and a straight road connects them. A fire f starts in the forest. Station a takes a measurement and notes that angle fab = 39°. Station b takes a measurement and notes that Angle fba = 74°. a) How far is it from each of the ranger stations to the fire? b) What is the distance from the road to the fire?

20.4) a) An angle o has a tangent of T. a) Make use of a right triangle and the Pythagorean theorem to solve for cos(o) and sin(o). b) Make use of the Pythagorean identity to do this problem again. Did you get the same answer?

21) a) From the viewpoint of an observer, the 'apparent size' of a sphere is 24° across at its widest. [Meaning if two rays share a common endpoint at the location of the observer, the largest possible angle these two rays can make while being tangent to the sphere is 24°]. The sphere's diameter is 4 feet. How far is it from the observer to the closest point of the sphere? b) Assuming the circumference of the earth is 24,900 miles, what would the 'apparent size of the earth' be for an astronaut 320 miles above the earth's surface?

22) Determine the Size of the Earth
Mauna Kea peak in Hawaii is 13,792 feet above sea level. From here the angle of depression from the horizontal to the horizon of the ocean is 2°5' (2 degrees 5 minutes). How big is the earth?

23) Determine the Size of the Earth
Eratosthenes an ancient Greek geometer estimated the size of the earth in the following manner. He noticed that on a certain day of the year, for a few moments, vertical objects cast no shadow. Wishing to determine the size of the earth, he traveled 492 miles north. On this same day of the year, at his new location he found that the smallest angle (from vertical) that shadows cast throughout the day is 7° 12'. From this information determine the size of the earth.

24) Determine the Size of the Earth
A sensor is able to determine the exact time of sunset. (defined as that time when the upper edge of the sun disappears beneath the horizon, from its vantage point). When the sensor is at ground level it detects that a sunset has occurred, and it is quickly raised to 5 feet above ground level. The sensor again detects the occurrence of a sunset 9.564 seconds later than before. Assuming the sensor is at a place and time where the sun passes directly overhead at noon time, calculate the size of the earth.

25) a) The chain of a bicycle transfers power from the front sprocket (diameter=8") to the rear sprocket (diameter=3"). If these sprocket centers were 9" apart, how long would the chain need to be? b) If this chain were to go around these sprockets in a figure 8, how long would the chain need to be?

26) The upper end of a 5 ft iron rod supports a 14" diameter pulley wheel at its center. The lower end of the rod is attached to a wall at a point 5 feet above the ground. The angle between the iron rod and the vertical (and the wall) is 43°. Two feet directly above the point where the iron rod connects to the wall, a rope connects to the wall and from there goes over the pulley wheel and then extends to the ground, where the rope stops. How long is the rope?

27.4) A circle has a radius of R. A chord of this circle has a length of L. What is the area of the smaller region bounded by the chord and the circle?

28) A circle of radius 6 has two chords sharing a common endpoint. One of these chords has a length of 2 the other has a length of 3. What is the area of the smallest possible region bounded by chords of this length and a circle of this diameter?

29) The centers of two circles, 12" and 10" in diameter, are separated by 9". What is the area of the region bounded by both circles? (Problem can be done with or without use of coordinates).

30.4) What are the orientations of the rays whose end points are at the origin, that are tangent to the parabola y=x^2+1?

31.4) From the roof of an apartment building, the angle of depression (from the horizontal) to the base of an office building is 51.5° and the angle of elevation (from the horizontal) to the top of the office building is 43.2°. If the office building is 847 ft high, how far apart are the two buildings and how high is the apartment building?

32.4) A boat is cruising on a straight course. A rocky point is sighted to the right at an angle of 28° with respect to the direction of travel. The boat continues on for 3.6 miles where the rocky point is spotted again to the right of the boat this time at an angle of 41° with respect to the direction of travel. How close will the boat come to the point?

Triangle Solutions

SSS (Side, Side, Side)
SAS (Side, Angle, Side)
ASA (Angle, Side, Angle)
AAS (Angle, Angle, Side)
HL (Hypotenuse, leg)

We know from classical geometry that if any of the above side/angle combinations of a triangle are given, the triangle is fully defined i.e. the other sides and angles not given are determined and can in principal be found out. In classical geometry we did not learn how to do this for the general case, though we did learn how to do it for a couple of triangle types, i.e. triangles whose angles are 45° 45°, 90°, and triangles whose angles are 30°, 60°, 90°).

In trigonometry, the limitation of having enough information so that a triangle is fully defined, but not be able to calculate the unknown sides and angles is overcome.

We now have enough tools (Algebra, Geometry and Trigonometry) to solve all triangles where sufficient information is given. In the Law of Cosines / Law of Sines section of this book we will derive more powerful tools to solve triangles that will make the job of doing it easier.

33) Prove: In a triangle, the altitude corresponding to any one of the bases (sides), is the length either of the other sides times the sine of the included angle. (the angle between the two sides)

34.4) Given the triangle asa=[47°,12,55°], determine all parts of this triangle which are not given, i.e. (sides, angles, and area).

35.4) a) Given the triangle sss=[3,5,7], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

36.4) Given the triangle sas=[7,15°,11], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

37) Given the triangle aas=[27°,71°,4], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

38) Given the triangle hl=[5,4], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

^^^^^^^^^^^^^^^^^^^^^^^^^^

5 Law of Cosines, Law of Sines

^^^^^^^^^^^^^^^^^^^^^^^^^^

In this section, we will make use of the Law of Cosines and the Law of Sines to solve triangles once again, i.e. determine the parts of a triangle not given, when any of the following are given.

SSS (Side, Side, Side)
SAS (Side, Angle, Side)
ASA (Angle, Side, Angle)
AAS (Angle, Angle, Side)
HL (Hypotenuse, leg)

The Law of Cosines and Law of Cosines are superior to the tools used in the last section to solve triangles. Using these tools makes the job easier.

In addition to solving triangles, we will make use of the Law of Cosines to derive a formula that gives the radius of the circle circumscribing a triangle in terms of the triangles sides.

Law of Cosines

The law of Cosines is an extension of the Pythagorean theorem. The Pythagorean theorem applies to right triangles. The law of cosines applies to any triangle. If two sides of a triangle and the angle between these two sides is given (SAS), the Law of Cosines gives the value, or the formula for the 3rd side. A,B and C are sides of a triangle, 'o' is the angle opposite C.

Law of Cosines ... C^2=A^2+B^2-2A*B*cos(o)

Notice that if o is a right angle, cos(o) is 0, and the Law of Cosines reduces to the Pythagorean theorem.

Exercise .. The derivation of the Law of Cosines follows. Before you look at this derivation, see if you can make use of coordinate geometry and your knowledge of trigonometry to derive the Law of Cosines for yourself.

Derivation of Law of Cosines

Consider a triangle whose sides have lengths A,B, and C, we refer to these sides as A, B and C respectively. o is the angle opposite side C. We place the vertex of the triangle opposite C at the origin, and we place the side B on the positive x axis. The side C therefore has endpoints whose coordinates are (B,0) and {A*cos(o),A*sin(o)}. By the distance formula we have the following.

$C^2 =$

$\{A*\cos(o)-B\}^2+\{A*\sin(o)-0\}^2 =$

$A^2*\cos^2(o)-2A*B*\cos(o)+B^2 + A^2\sin^2(o) =$

$A^2\cos^2(o)+A^2\sin^2(o)+B^2-2A*B*\cos(o) =$

$A^2\{\cos^2(o)+\sin^2(o)\}+B^2-2A*B*\cos(o) =$

$A^2(1)+B^2-2A*B*\cos(o) =$

$A^2+B^2-2A*B*\cos(o)$

therefore

$C^2 = A^2+B^2-2A*B*\cos(o)$ <------ Law of Cosines

Derivation Complete

<u>Law of Cosines Problem Set</u>

1) Prove the law of cosines, $C^2=A^2+B^2-2A*B*\cos(o)$.

2) A triangle has a side of 3 and another side of 4. The angle between these sides is 25°. a) Determine the length of third side of this triangle. b) What is the area of this triangle?

3) A triangle has sides of length 2,3 and 4. a) What is the cosine of the angle opposite the side whose length is 2? b) What is the sine of the angle opposite the side whose length is 2? c) What is the sine of the angle opposite the side whose length is 3?

4) A triangle has sides of length A, B and C. a) What is the cosine of the angle opposite the side whose length is A? b) What is the sine of the angle opposite the side whose length is C?

5) Prove: The altitude of a triangle corresponding to one of its sides is the product of length of either of the other sides and the sine of the included angle.

6) a) A triangle has sides of length A, B and C. Derive a formula for its area. b) Make use of the formula you derived in part 'a' to derive Heron's formula, i.e.

$$\text{Area triangle} = \sqrt{S(S-A)(S-B)(S-C)} \text{ where } S= \frac{A+B+C}{2}$$

Law of Sines

Law of sines $$\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$$

This identity applies to all triangles.

Law of Sines Derivation

A,B and C are the sides of an arbitrary triangle T. a,b,c are the angles opposite these sides. a is opposite A, b is opposite B, c is opposite C.

Area of T = 1/2 * triangle side * altitude that corresponds to side

altitude of T corresponding to side A is .. B*sin(c) or C*sin(b)
altitude of T corresponding to side B is .. C*sin(a) or A*sin(c)
altitude of T corresponding to side C is .. A*sin(b) or B*sin(a)

therefore

Area of T = 1/2 * A * {B*sin(c)} -> 2*Area of T = A*B*sin(c)
Area of T = 1/2 * B * {C*sin(a)} -> 2*Area of T = B*C*sin(a)
Area of T = 1/2 * C * {A*sin(b)} -> 2*Area of T = C*A*sin(b)

therefore

A*B*sin(c) = B*C*sin(a) = C*A*sin(b) ->

$$\frac{A*B*\sin(c)}{A*B*C} = \frac{B*C*\sin(a)}{A*B*C} = \frac{C*A*\sin(b)}{A*B*C} \quad ->$$

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$$ <--- Law of Sines (Derivation Complete)

Example Problem

A triangle has a side of 3, a side of 5, the angle between these two sides is 17° . Calculate all other sides and angles of this triangle.

Solution

Let A,B and C be the sides of this triangle, let a be the angle opposite A, let b be the angle opposite B, let c be the angle opposite C.

Let A=3, let B=5, let c=17°

By the law of cosines

C=
sqrt{3^2 + 5^2 - 2*3*5*cos(17°)} =
sqrt{9 + 25 - 30cos(17°)} = 2.3045

By the law of sines, $\dfrac{\sin(17°)}{2.3045} = \dfrac{\sin(a)}{3} = \dfrac{\sin(b)}{5}$

I) Therefore $\sin(a)= \dfrac{3\sin(17°)}{2.3045} = 0.38061$

II) Therefore $\sin(b) = \dfrac{5\sin(17°)}{2.3045} = 0.63435$

In trying to solve for 'a' and 'b' in equations I and II it is noted there are an infinite number of angles whose sine is any particular value. Equations I and II each have an infinite number of solutions. Only one of these solutions is the one we are looking for. How do we determine the right solution? See if you can determine this for yourself before you continue. --Stop--

Any angle in a triangle is between 0° and 180°. Therefore all angles we are looking for are between 0° and 180°. Therefore unless a=90° {i.e. sin(a)=1} there are two possible angles between 0° and 180° that 'a' can be, one less than 90° and one greater than 90°. [Draw a diagram, take a few moments to convince yourself this is true].

In a triangle all angles can be less than 90°, however only one angle can be greater than 90° (else two angles added together would be greater than 180°). From classical geometry we know A<B implies a<b, therefore 'a' is guaranteed to be less than 90°. Therefore we solve for 'a' (the angle opposite the shortest side) first and assign to it the solution of sin(a)=0.3861 that is less than 90° and greater than 0°. We would then assign to 'b' the value which makes the sum of the angles of the triangle to equal 180° therefore ..

$b = 180° - a - c \rightarrow$
$b = 180° - 22.371° - 17° = 140.629°$.

We summarize our results as follows, The sides of this triangle are A=3, B=5, C=2.3045. The angles of this triangle are a=22.371°, b=140.629° and c=17°.

To help ensure these calculations are correct, we can do a Law of Sines check on the triangle we calculated.

Law of Sines Test
We apply the law of sines as follows to the calculated triangle to determine if it "obeys" the Law of Sines.

$$\frac{\sin(a)}{A} \overset{?}{=} \frac{\sin(b)}{B} \overset{?}{=} \frac{\sin(c)}{C} \rightarrow$$

$$\frac{\sin(22.371°)}{3} \overset{?}{=} \frac{\sin(140.63°)}{5} \overset{?}{=} \frac{\sin(17°)}{2.3045} \rightarrow$$

$$0.12687 \overset{?}{=} 0.12687 \overset{?}{=} 0.12687$$

The law of sines does apply to the triangle we calculated, it appears our calculations are correct.

Law of Cosines, Law of Sines Problem Set

1) Prove: In a triangle, the altitude corresponding to any one of the bases (sides), is the length either of the other sides times the sine of the included angle. (the angle between the two sides)

2) Prove the law of sines, $\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$

3) a) Given the triangle sss=[3,5,7], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

4) Given the triangle sas=[7,15°,11], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

5) Given the triangle asa=[47°,12,55°], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

6) Given the triangle aas=[27°,71°,4], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

7) Given the triangle HL=[5,4], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

8) Given the triangle sss=[A,B,C]; a) Determine the angles of this triangle. b) Determine the area of this triangle.

9) Write a calculator or computer program to determine the area and all sides and angles of a triangle, when given a) SSS; b) SAS; c) ASA; d) AAS; e) HL

Law of Cosines, Law of Sines Special Topic

In this section we make use of the Law of Cosines, to derive a formula giving the radius of a circle inscribing a triangle, in terms of lengths of the sides of the triangle.

The Picture
A triangle T has a side S. The angle opposite S is o. Circle C circumscribes T. The radius of C is R. The center of C is c. The end points of S are s' and s". p is the point where the perpendicular from c to S, meets S.

R=S/{2sin(o)} derivation outline: By classical Geometry theorem, angle s' c s" = 2 * angle o. From here you can get that angle s' c p = angle o. From here you can derive the theorem sin(o)=(S/2)/R implying that R=S/{2sin(o)}.

Note: The formula R=S/{2sin(o)} is always true. However the (preceeding) derivation outline doesn't show the truthfulness of this formula if S is a hypotenuse of a right triangle. This proof is not difficult and is left to the student.

Example Problem) A triangle has sides A,B,C. Angle a is opposite A. Angle b is opposite B. Angle c is opposite C. Calculate the sine of c, the angle opposite C, in terms of A,B,C. Make use of this result and R=S/{2sin(o)} to derive an expression for the radius of a circle inscribing a triangle in terms of its sides A,B,C.

By the Law of Cosines we have

C^2=A^2+B^2-2*A*B*cos(c) ->

2*A*B*cos(c)=A^2+B^2-C^2 ->

```
          A^2+B^2-C^2
cos(c)= ------------- ->
            2*A*B
```

```
                         {A^2+B^2-C^2 }^2
sin^2(c)= 1-cos^2(c) = 1-{------------}   =
                         {   2*A*B    }


   (A^4)+(B^4)+(C^4)+(2*A^2*B^2)-(2*A^2*C^2)-(2*B^2*C^2)
1- ------------------------------------------------------ =
                          4*A^2*B^2


4*A^2*B^2   (A^4)+(B^4)+(C^4)+(2*A^2*B^2)-(2*A^2*C^2)-(2*B^2*C^2)
--------- - ------------------------------------------------------ =
4*A^2*B^2                        4*A^2*B^2


(4*A^2*B^2)-(2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^2)+(B^4)+(C^4)
------------------------------------------------------------------ =
                            4*A^2*B^2


(2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^2)+(B^4)+(C^4)
-----------------------------------------------------  .. therefore
                      4*A^2*B^2



          (2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^2)+(B^4)+(C^4)
sin^2(c)= -----------------------------------------------------  ->
                               4*A^2*B^2


            {(2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^2)+(B^4)+(C^4)}
sin(c)= sqrt{-----------------------------------------------------}
            {                    4*A^2*B^2                        }
```

or

```
        sqrt{(2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^2)+(B^4)+(C^4)}
sin(c)= -----------------------------------------------------------
                                2*A*B
```

[Where C = S and c = o, R=S/{2sin(o)} becomes R=C/{2sin(c)}]

we substitute the equation sin(c)= .. (which is just above the previous line) into R=C/{2sin(c)}

This gives us

```
                              C
R= -----------------------------------------------------------------
          sqrt{(2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^2)+(B^4)+(C^4)}
     2 * -------------------------------------------------------------
                             2*A*B

                         A*B*C
R= -----------------------------------------------------------
   sqrt{(2*A^2*B^2)+(2*A^2*C^2)+(2*B^2*C^2)+(A^4)+(B^4)+(C^4)}


                             A*B*C
R= ---------------------------------------------------
   sqrt{2(AB)^2 + 2(AC)^2 + 2(BC)^2 + A^4 + B^4 + C^2}
```

Example Problem Completed

It is possible to exchange any of the sides of the triangle A,B, or C with each other. Doing this would not change the radius of the circle that circumscribes the triangle. Given this, if the above equation has been derived correctly, exchanging any of the variables A,B,C with each other will not change the value of R. For example, renaming all A's in the equation to C, and renaming all C's in the equation to A should would not change the equation or its value. If exchanging variables in an equation does not change the value of the equation, we say the equation is symmetric with respect to those variables. Whenever a situation described by an equation symmetric, the equation will be symmetric, and visa versa. If not, it has been derived incorrectly.

Law of Cosines, Law of Sines - Special topic Problem Set

1) Prove: The radius of the circle that circumscribes a triangle is the length of any of its sides divided by twice the sine of the angle opposite that side, i.e. R=S/{2sin(o)}. [This is two separate proofs, one proof where S is not a hypotenuse of a triangle, do this proof first. The other proof is where S is the hypotenuse of a triangle, do this proof second].

2) Make use of the theorem R=S/{2sin(o)} to derive the Law of Sines.

3) a) Derive an equation that will give you the sine of an angle of a triangle in terms of the sides of the triangle. b) Substitute this equation into the theorem R=S/{2sin(o)} thereby deriving an equation that gives the radius of a circle that circumscribes a triangle, in terms of the lengths of the sides of the triangle.

2) A triangle has sides 4,5,6. What is the radius of the circle that circumscribes this triangle?

4) Write a calculator or computer program whose inputs are the sides of a triangle. The output of this program is the radius of the circle which circumscribes this triangle.

Bonus Problems

1) Where A,B,C are sides of a triangle and a,b,c are the corresponding (opposite) angles, prove

$$\frac{\cos(a)}{A} + \frac{\cos(b)}{B} + \frac{\cos(c)}{C} = \frac{A^2+B^2+C^2}{2A*B*C}$$

2) Make use of the law of cosines to prove .. In a parallelogram, the sum of every 'diagonal squared' is equal to the sum of every 'side squared'.

3.5.4) Prove: The area of any quadrilateral is equal to the one half of the product of (the lengths of) its diagonals times the sine of the angle o they form. (where o is the acute angle formed by the diagonals or is 90° if d1 and d2 are perpendicular).

4.5.4) Where A,B are the sides of a triangle, and a,b are the corresponding (opposite) angles,

prove: $$\frac{A-B}{A+B} = \frac{\sin(a)-\sin(b)}{\sin(a)+\sin(b)}$$

This (last) identity is remarkably similar to the 'Law of Tangents'. The student is asked to prove the Law of Tangents in problem 35.15 in the 'General Problem Set 2 (Advanced)' section.

^^^^^^^^^^^^^^^^

6 Deriving Identities Geometrically

^^^^^^^^^^^^^^^^

Preliminary Problems

The main development of identities in this book will be analytic, not geometric. Analytic development has the advantage that such proofs apply to all relevant angle sizes automatically. Classical geometry trigonometry proofs naturally apply to angles of limited scope, i.e. greater than 0° but less than 90° for example. If one wants to prove a trigonometric identity using a classical geometry proof for all possible angle sizes it takes more work, several cases have to be considered. In spite of the fact that geometric proofs can be more cumbersome if the proof is to apply to all angle sizes, they are worthwhile. They show a different way of doing things and add to the student's problem solving experience.

The reason the problems of this preliminary section are given is to help prepare the student to prove the cosine and sine addition and subtraction formulas geometrically. These preliminary problems are wonderful problems in their own right. There are two methods of doing the problems of this section. The first makes use of trigonometric ratios to calculate the lengths of all unknown sides. The second makes use of trigonometric ratios to find the length of the first sides and the Pythagorean identity to calculate the length of the last side. Both of these methods yield the same result of course, but the results are in different forms. Until directed otherwise, is important the problems of this section be done using the method that uses trigonometric ratios exclusively. This method yields a form of the answer that is simpler and more useful. The student will be asked to solve problems 4, 5 and 6 using both methods, and then verify both methods yield the same answer.

Problem Set

1*) o has a measure of 17 degrees and is opposite one of the legs of a right triangle. The leg adjacent o has a length of 4. What are the lengths of the other sides of this triangle?

2) o has a measure of 31 degrees and is opposite one of the legs of a right triangle. The leg opposite o has a length of 9. What are the lengths of the other sides of this triangle?

3) o has a measure of 61 degrees and is opposite one of the legs of a right triangle. The Hypotenuse of this triangle has a length of 7. What are the other sides of this triangle?

4) o is an angle opposite one of the legs of a right triangle. The leg adjacent o has a length of k. a) What are the lengths of the other sides of this triangle? b) Solve this problem again, making use of the Pythagorean identity to calculate the last side. Simplify and verify this is the same answer the first method yields.

5) o is an angle opposite one of the legs of a right triangle. The leg opposite o has a length of k. a) What are the lengths of the other sides of this triangle? b) Solve this problem again, making use of the Pythagorean identity to calculate the last side. Simplify and verify this is the same answer the first method yields.

6) o is an angle opposite one of the legs of a right triangle. The hypotenuse of this triangle has a length of k. a) What are the lengths of the other sides of this triangle? b) Solve this problem again, making use of the Pythagorean identity to calculate the last side. Simplify and verify this is the same answer the first method yields.

7) AB is a segment. D is a point on AB and CD is a segment perpendicular to AB. Take note of the two right triangles ADC and BDC sharing the common leg CD. Angle a is the angle of triangle ADC opposite the leg CD. Angle c is the angle of triangle BDC opposite the leg DB. The leg AD of triangle ADC has a length of k. What are the lengths of all sides of triangles ADC and BCD?

8) AB is a horizontal segment. Point C is directly above A and point D is directly below B. c is the angle of triangle CAB which is opposite leg AB. d is the angle of triangle DBA opposite leg AB. Leg DB of triangle DBA has a length of k. What are the lengths of all sides of triangles CAB and DBA?

Geometrically Proving Cosine Addition and the Sine Subtraction Formulas and Other Selected Trigonometric Identities

In this section, several important trigonometric identities are proven making use of classical geometry. The proofs offered here are incomplete in that the values of the angles for which these proofs are valid are restricted. Using classical geometry it is possible but cumbersome to prove these identities for the general case. Following this section these identities will be proven again using either Coordinate Geometry or Trigonometric techniques, these proofs will be complete.

Deriving Identities Geometrically Problem Set

1*) a) Make use of a right triangle to prove that cos(90°-o)=sin(o);
b) Make use of a right triangle to prove that sin(90°-o)=cos(o).

2*) Make use of the circle definition of cos(o) to geometrically prove that cos(-o)=cos(o) (0°<o<90°). Then take a few moments, to prove or to convince yourself that cos(-o)=cos(o) is true for all angles o, (-infinity < o < infinity).

3) Make use of the circle definition of sin(o) to prove that sin(-o)=-sin(o) (0°<o<90°). Then take a few moments, to prove or to convince yourself that sin(-o)=sin(o) is true for all angles o, (-infinity < o < infinity).

4.6.2) Make use of Classical Geometry to prove the cosine addition formula, i.e. prove cos(u+v)=cos(u)cos(v)-sin(u)sin(v). (0°<u<180°), (0°<v<90°) and (0°<u+v<90°).

5.6.2) Make use of Classical Geometry to prove the sine subtraction formula, i.e. prove sin(u-v)=sin(u)cos(v)-cos(u)sin(v) where (0°<u<180°) (0°<v<90°) (0°<u-v<90°)

In the next section the cosine addition and subtraction formulas, along with the sine addition addition and subtraction formulas will be proven in a way such that the angle restrictions will be unnecessary.

^^^^^^^^^^^^^^^^^^^^^

7 Deriving Identities Analytically

^^^^^^^^^^^^^^^^^^^^^

In this section, the cosine, sine and tangent addition and subtraction identities and other important identities are derived. All identities derived in the previous section using classical Geometry are derived here again. This section is main development of the beginning identities of this book. Trigonometry identities were derived using (near) classical geometry techniques in the previous section to demonstrate that identities can be derived using classical Geometry and to give students an opportunity to have wider problem solving experience.

The following three identities are postulated, no proof is offered.

```
  cos(0°)= 1  .  .  (Postulate)  . . .  I2) Identity 2
 sin(90°)= 1  .  . (Postulate)  . . .  I3) Identity 3
cos(180°)=-1  .  . (Postulate)  . . .  I4) Identity 4
```

Take the time to convince yourself these postulates are true.

Proof that sin(0°)=0

```
I1 the Pythagorean identity ->
cos^2(0°)^2+sin^2(0°)=1
sin^2(0°)=1-cos(0°)^2 ->  ... applying I2
sin^2(0°)=1-1^2 ->
sin^2(0°)=1-1 ->
sin^2(0°)=0
sin(0°)=0   <---- I5 (Identity 5)
```


Proof that cos(90°)=0

```
I1 the Pythagorean identity ->
cos^2(90°)+sin^2(90°)=1 ->
cos^2(90°)=1-sin^2(90°) -> ... applying I3
cos^2(90°)=1-1^2 ->
cos^2(90°)=1-1 ->
cos^2(90°)=0 ->
cos(90°)=0    <----- I6 (Identity 6)
```

Proof that sin(180°)=0
I1 the Pythagorean Identity ->
sin^2(180°)+cos^2(180°)=1 ->
sin^2(180°)=1-cos^2(180°) -> ... applying I4
sin^2(180°)=1-(-1)^2 ->
sin^2(180°)=1-1
sin^2(180°)=0 ->
sin(180°)=0   <---- I7 (Identity 7)

Table of Preliminary Identities

I1) cos^2(o)+sin^2(o)=1
I2) cos(0°)=1,   I5) sin(0°)=0
I6) cos(90°)=0,   I3) sin(90°)=1
I4) cos(180°)=-1 I7) sin(180°)=0

Cosine Subtraction Identity
Coordinate Geometry Derivation

cos(a-b)=cos(a)cos(b)+sin(a)sin(b) .. cosine subtraction formula

The sine and cosine addition and subtraction formulas can all be derived using classical geometry, from there trigonometry equation manipulation may be needed to put equations into standard form. Multiple classical geometry derivations of a single addition or subtraction formula are required because of the 4 quadrants of the Cartesian coordinate system. The magic of applying algebra to a geometry problem allows a single coordinate geometry proof to prove all possible cases using only one proof.

Here coordinate geometry is used to derive the cosine subtraction formula, the remaining subtraction and addition formulas are derived making use of this formula, using trigonometry (algebraic) techniques. We derive the cosine subtraction formula first because this route doesn't require knowledge of the identities cos(-o)=cos(o) and sin(-o)=-sin(o) before they are derived.

Remember: {cos(a),sin(a)} is (defined as) the point where the ray whose end point is at the origin and whose orientation is angle 'a' intersects the unit circle centered at the origin.

Consider the following picture. A unit circle is centered at the origin and the segment AB is a chord of this circle named C. A is located at the point {cos(a),sin(a)} and B is located at the point {cos(b),sin(b)}.

If this segment is moved, i.e. rotated by an amount of angle -a around the circle, such that its endpoints are still on the circle. In its new position, The coordinates of its end point A is {cos(0°),sin(0°)} or (1,0) and the coordinates of its end point B is {cos(b-a),sin(b-a)}.

To derive the cosine subtraction formula, we use the distance formula to find the lengths of the chord C before and after it is moved and set these two lengths equal to each other. From here we solve for the term cos(b-a).

Take a few moments to study and assure yourself you understand how this derivation works. Then derive the cosine subtraction formula cos(a-b)=cos(a)cos(b)-sin(a)sin(b) on your own if you can. A derivation of the cosine subtraction formula follows.

Proof that cos(a-b)=cos(a)cos(b)+sin(a)sin(b)

Consider the following picture. A unit circle is centered at the origin and the segment AB is a chord of this circle named C. A is located at the point {cos(a),sin(a)} and B is located at the point {cos(b),sin(b)}.

If this segment is moved, i.e. rotated by an amount of angle -a around the circle, such that its endpoints are still on the circle. In its new position, The coordinates of its end point A is now located at the point A'{cos(0°),sin(0°)}, which according to I2 and I5 is A'(1,0), and the coordinates of its end point B is now located at B'{cos(b-a),sin(b-a)}.

We now calculate the length of C before it was moved

AB^2= ... Applying the distance formula we have
{cos(b)-cos(a)}^2+{sin(b)-sin(a)^2} =
cos^2(b)-2cos(b)cos(a)+cos^2(a) + sin^2(b)-2sin(b)sin(a)+sin^2(a)=
{cos^2(b)+sin^2(b)}+{cos^2(a)+sin^2(a)}-2cos(b)cos(a)-2sin(b)sin(a)=
1 + 1 -2cos(b)cos(a) -2sin(b)sin(a) =
2 -2cos(b)cos(a) -2sin(b)sin(a)

We now calculate the length of C after it was moved.

A'B'^2= ... Applying the distance formula we have
(cos(b-a)-1)^2+(sin(b-a)-0)^2 =
cos^2(b-a)-2cos(b-a)+1^2 + sin^2(b-a) =
{cos^2(b-a)+sin^2(b-a)}-2cos(b-a)+1=
1-2cos(b-a)+1 =
2-2cos(b-a)

We assume here that merely moving a chord, does not change its length therefore length of C before moving it = length of C after moving it ->

AB=A'B' ->

AB^2=A'B'^2 ->

2-2cos(b)cos(a)-2sin(b)sin(a) = 2-2cos(b-a) ->

cos(b-a)=cos(b)cos(a)+sin(b)sin(a) <- I8 cosine subtraction formula

Proof Complete

Addition and Subtraction Formulas (cos, sin, tan) continued

Definitions:

If f(-x)=f(x), f is an even function

If f(-x)=-f(x), f is an odd function

Now we derive some more preliminary identities necessary to derive the cosine addition formula.

Proof that cos(o) is an even function i.e. proof that cos(-o)=cos(o).

cos(-o)=

cos(0°-o)= ... applying I8 we get

cos(0°)cos(o)+sin(0°)sin(o)= ... applying I2,I5 we get

(1)cos(o)+(0°)sin(o)=

cos(o)

therefore

cos(-o)=cos(o) <---- I9   cosine is an even function

Now we wish to prove that sin is an odd function. The next two identities presented and proven are used to help accomplish this. These next two identities are also important in their own right.

Proof that cos(90°-o)=sin(o)

cos(90°-o)=   ... applying I8 we get

cos(90°)cos(o)+sin(90°)sin(o)= ... applying I3 and I6 we get

(0)cos(o)+(1)sin(o)=

sin(o)

Therefore cos(90°-o)=sin(o) <--- I10

Proof that cos(o-180°)= -cos(o)

cos(o-180°)= ... applying I8 we get
cos(o)cos(180°)+sin(o)sin(180°)= ... applying I4 and I7 we get
cos(o)(-1)+sin(o)(0°)=
-cos(o)

Therefore

cos(o-180)= -cos(o) <--- I11 ->
cos(o)= -cos(o-180) <--- I11'

Proof that sin is an odd function, i.e.
proof that sin(-o)= -sin(o).

sin(-o)= ... applying I10 we get
cos{90°-(-o)}=
cos(90°+o)= ... applying I11' we get
-cos{(90°+o)-180°}
-cos(o-90°)= ... applying I9 we get
-cos(90°-o)= ... applying I10 we get
-sin(o)

Therefore

sin(-o)= -sin(o) <--- I12 sin is an odd function

Proof of cosine addition formula, i.e.
prove that cos(a+b)=cos(a)cos(b)-sin(a)sin(b)

cos(a+b)=
cos{a-(-b)}= ... applying I8 we get
cos(a)cos(-b)+sin(a)sin(-b)= ... applying I9, I12 we get
cos(a)cos(b)+sin(a){-sin(b)}=
cos(a)cos(b)-sin(a)sin(b)

therefore

cos(a+b)=cos(a)cos(b)-sin(a)sin(b) <--- I13 cosine addition formula

Cosine Angle Addition/Subtraction Formula Problems

1) Given the following, determine cos(a+b)=? and cos(a-b)=?
  a) cos(a)=3/4, sin(a) is positive, cos(b)=1/3, sin(b) is positive
  b) cos(a)=1/9, sin(a) is negative, sin(b)=-3/7, cos(b) is positive
  c) sin(a)=-1/2, cos(a) is negative, sin(b)=3/4, cos(b) is negative

2) Without making use of calculator trig functions, find cos(75°)
3) Without making use of calculator trig functions, find cos(15°)
4) Without making use of calculator trig functions, find cos(105°)

Note: [The first point defining the rays below is their endpoint].

5) Calculate the cosine of the acute angle defined by rays
  [(0,0):(1,3)] and [(0,0):(1,5)]

6) Calculate the cosine of the acute angle defined by rays
  [(0,0):(-1,2)] and [(0,0):(2,7)]

7) a) State the cosine addition and subtraction formulas from memory. b) Write the cosine addition and subtraction formulas from memory.

Continuing with trigonometry identity development

Proof that sin(90°-o)=cos(o)

I10 ->
sin(o)=cos(o-90°) -> substituting 90-o into o we get
sin(90°-o)=cos({90°-o}-90°) ->
sin(90°-o)=cos(-o) -> applying I9 we get
sin(90°-o)=cos(o) <--- I14

Derive the sin addition formula, i.e.
prove that sin(a+b)=sin(a)cos(b)+sin(b)cos(a)

sin(a+b)=   ... applying I10
cos{90°-(a+b)}=
cos{(90°-a)-b}=   ... applying I8 we get
cos(90°-a)cos(b)+sin(90°-a)sin(b)=   ... applying I10 and I14 we get
sin(a)cos(b)+cos(a)sin(b)

therefore

sin(a+b)=sin(a)cos(b)+cos(a)sin(b) <--- I15 sine addition formula

Derive the sin subtraction formula, i.e.
prove that sin(a-b)=sin(a)cos(b)-sin(b)cos(a)

sin(a-b)=
sin(a+{-b})=   ... applying I15
sin(a)cos(-b)+sin(-b)cos(a)= ... applying I9 and I12
sin(a)cos(b)+{-sin(b)}cos(a)=
sin(a)cos(b)-sin(b)cos(a)
therefore

sin(a-b)=sin(a)cos(b)-cos(a)sin(b) <-- I16 sine subtraction formula

----

Sine Angle Addition/Subtraction Formula Problems

1) Given the following, determine cos(a+b)=? and cos(a-b)
  a) cos(a)=1/3, sin(a) is positive, cos(b)= 1/2, sin(b) is positive
  b) cos(a)=2/5, sin(a) is positive, sin(b)= 2/3, cos(b) is negative
  c) sin(a)=3/4, cos(a) is negative, sin(b)=-1/2, cos(b) is negative

2) Without making use of calculator trig functions, find sin(75°)
3) Without making use of calculator trig functions, find sin(15°)
4) Without making use of calculator trig functions, find sin(105°)

Note: The first point defining the rays below is their end point.

5) Calculate the sine of the acute angle defined by the rays
  [(0,0):(3,11)] and [(0,0):(2,9)]

6) Calculate the sine of acute angle defined by the rays
  [(0,0):(3,2)] and [(0,0):(1,-7)]

7) a) State the sine addition and subtraction formulas from memory.
  b) Write the sine addition and subtraction formulas from memory.

Definition tan(o): tan(o)= sin(o)/cos(o)

Derivation of the tangent addition formula

tan(a+b)= ... by the definition of tan(o)

```
 sin(a+b)
---------- =  ... making use of I13 and I15
 cos(a+b)
```

$$\frac{\sin(a)\cos(b)+\sin(b)\cos(a)}{\cos(a)\cos(b)-\sin(a)\sin(b)} = \quad \text{... dividing numerator and denominator by cos(a)cos(b)}$$

$$\frac{\dfrac{\sin(a)\cos(b)}{\cos(a)\cos(b)}+\dfrac{\sin(b)\cos(a)}{\cos(a)\cos(b)}}{\dfrac{\cos(a)\cos(b)}{\cos(a)\cos(b)}-\dfrac{\sin(a)\sin(b)}{\cos(a)\cos(b)}} = \quad \text{... this simplifies to}$$

$$\frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$$

therefore

$$\tan(a+b)=\frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)} \quad \text{<-----} \quad \text{I17 tangent addition formula}$$

In exercise sections, the student will be asked to prove

$$\tan(a-b)=\frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)} \quad \text{<-----} \quad \text{I18 tangent subtraction formula}$$

<u>Tangent Angle Addition/Subtraction Formula Problems</u>

1) Given the following, determine tan(m+n)=? and tan(m-n)
  a) tan(m)= 2, tan(n)=3; b) tan(m)= -1, tan(n)=7.

2) Without making use of calculator trig functions, find tan(15°)
3) Without making use of calculator trig functions, find tan(165°)
4) Without making use of calculator trig functions, find tan(300°)

5) Calculate the tangent of the acute angle defined the intersection of the lines, y=2x and y=3x

6) Prove: If a ray is oriented at an angle of o, then tan(o) equals the slope of the ray.

Note: The first point defining the ray below is its endpoint.

7) Calculate the tangent of acute angle defined by rays [(0,0):(1,-2)] and [(0,0):(-5,7)]

8) a) State the tangent addition and subtraction formulas from memory. b) Write the tangent addition and subtraction formulas from memory.

Double, Squared and Half Angle Formulas

```
Prove cos(2o)=cos^2(o)-sin^2(o) = 2cos^2(o)-1 = 1-2sin^2(o)

cos(2o)=
cos(o+o)=  ... applying I13 we get
cos^2(o)-sin^2(o)=

therefore

cos(2o)= cos^2(o)-sin^2(o) <--- I19a  cosine double angle formula

I19a ->
cos(2o)=cos^2(o)-sin^2(o) ->  ... applying I1 we get
cos(2o)=cos^2(o)-[1-cos^2(o)] ->
cos(2o)=2cos^2(o)-1  <--- I19b cosine double angle formula

I19a ->
cos(2o)=cos^2(o)-sin^2(o) ->  ... applying I1 we get
cos(2o)=[1-sin^2(o)]-sin^2(o) ->
cos(2o)=1-2sin^2(o)  <--- I19c  cosine double angle formula

---------------------------
---------------------------

                1+cos(2o)
Prove cos^2(o)= ---------
                    2

I19b ->
cos(2o)=2cos^2(o)-1 ->

           1+cos(2o)
cos^2(o)= ---------   <--- I20 cosine squared formula
              2

---------------------------
---------------------------
```

```
                1-cos(2o)
Prove sin^2(o)= ---------
                    2

I19c ->
cos(2o)=1-2sin^2(o) ->


          1-cos(2o)
sin^2(o)= ---------  <--- I21 sine squared formula
              2

------------------------
------------------------

                         {1+cos(o)}
Derive cos(o/2)= +/- sqrt{--------}
                         {   2    }

I20 ->

          1+cos(2a)
cos^2(a)= --------- =  ... taking the square root of both sides
              2


                [1+cos(2a)]
cos(a)= +/- sqrt[---------] = ... substituting o/2 into 'a'
                [    2    ]


   (o)          {1+cos(o)]
cos(-)= +/- sqrt{--------]  <--- I22 cosine half angle formula
   (2)          {   2    ]


---------------------------
---------------------------

                         {1-cos(o)}
Derive sin(o/2)= +/- sqrt{--------}
                         {   2    }

I21 ->

          1-cos(2a)
sin^2(a)= --------- =  ... taking the square root of both sides
              2
```

```
                [1-cos(2a)]
sin(a)= +/- sqrt[---------] = ... substituting o/2 into 'a'
                [    2    ]


   (o)          {1-cos(o)]
sin(-)= +/- sqrt{--------]  <--- I23 sine half angle formula
   (2)          {   2    ]


---------------------------
---------------------------

          (o)          {1-cos(o)}
Derive tan(-)= +/- sqrt{--------}
          (2)          {1+cos(o)}


definition of tangent ->

        sin(a)
tan(a)= ------  ->  substituting o/2 into a we get
        cos(a)


            (o)
         sin(-)
   (o)      (2)
tan(-)= --------  -> applying I22 and I23 we get
   (2)      (o)
         cos(-)
            (2)


                  {1-cos(o)}
          +/- sqrt{--------}
   (o)            {   2    }
tan(-) = ---------------------  ->
   (2)            {1+cos(o)}
          +/- sqrt{--------}
                  {   2    }


   (o)           {1-cos(o)}
tan(-) = +/- sqrt{--------}    <--- I24 tangent half angle formula
   (2)           {1+cos(o)}
```

Double, Squared and Half Angle Formulas Problem Set

Before doing this problem set, you should have reviewed the derivations of the cosine, sine and tangent half angle formulas beginning with I19, the cosine double angle formula. You should not memorize these formulas but you should know how to derive them. In the following problem set when you need any double angle formulas, squared formulas or half angle formulas, derive them for yourself.

1) cos(o) is 1/5, what are possible value(s) of cos(2o)?

2) cos(o) is 1/5, what are possible value(s) of cos^2(o)?

3) cos(o) is 1/5, what are possible value(s) of cos(o/2)?

4 & 5 preparation) cos(x)= 0.5 : a) |sin(x)|=?; |tan(x)|=?

4 & 5 preparation) tan(x)=3 : |cos(x)| =?; |sin(x)| =?

4) A ray with an orientation of o (0°<o<90°) lies in the 1st quadrant and has a slope of 2. Make use of the tangent half angle formula help you calculate what slope a ray with an orientation of o/2 has?

5) A ray with an orientation of o (180°<o<270°) lies in the 3rd quadrant and has a slope of 2. Make use of the tangent half angle formula help you calculate what slope a ray with an orientation of o/2 has?

6) Derive a sine half angle formula that makes use of sine function(s) not a cosine function.

As has been mentioned before, each time you come to an identity and its proof you should try to prove the identity on your own without looking at the proof. If you are unable to prove this identity, you should look at the proof just long enough to get the needed hints, and then you should prove the identity on your own without looking. By the time you get to this place in the book, it is expected you have proved all identities up to this point. If not you should go back to the beginning of this chapter and do it now. Otherwise you are unprepared to continue.

Deriving Identities Analytically Problem Set

1) a) Give the triangle definition for $\sin(o)$ and $\cos(o)$; b) Give the circle definition for $\sin(o)$ and $\cos(o)$; c) Draw a picture illustrating each definition, and from these pictures, explain the connection between the triangle and circle definitions.

2) The following identities are important, they should be learned or memorized. Look them over and then without looking a) Write the list of these identities from memory. b) Verbally state each of these identities from memory.
   a) Pythagorean Identity
      $\cos^2(o)+\sin^2(o)= 1$; $1-\cos^2(o)=\sin^2(o)$; $1-\sin^2(o)=\cos^2(o)$
   b) $\cos(-o)= \cos(o)$ cosine is an even function
   c) $\sin(-o)= -\sin(o)$ sine is an odd function
   d) $\cos(o \pm 180°)= -\cos(o)$: $\sin(o \pm 180°)= -\sin(o)$
   e) $\cos(90°-o)=\sin(o)$; $\sin(90°-o)=\cos(o)$
   f) $\cos(a+b)= \cos(a)\cos(b)-\sin(a)\sin(b)$ cosine addition formula
   g) $\cos(a-b)= \cos(a)\cos(b)+\sin(a)\sin(b)$ cosine subtraction formula
   h) $\sin(a+b)= \sin(a)\cos(b)+\cos(a)\sin(b)$ sine addition formula
   i) $\sin(a-b)= \sin(a)\cos(b)-\cos(a)\sin(b)$ sine subtraction formula
   j) $\tan(a+b)= \{\tan(a)+\tan(b)\}/\{1-\tan(a)\tan(b)\}$ tan addition formula
   k) $\tan(a-b)= \{\tan(a)-\tan(b)\}/\{1+\tan(a)\tan(b)\}$ tan subtract formula

Fork in the Road
Do one of the following options.
I) Regular Option: do 3 Regular Option)
II) Honors Option: do 3 Honors Option)

If you feel you are up to it, the Honors Option is the recommended option. Each of these options have students deriving identities they should have already derived. (Students should be proving all theorems in the book at the time they are presented). Proving these identities once again will help cement each of these fundamental identities and their proofs and the development that leads up to each of these proofs into the students consciousness.

3 Regular Option)

You may consider the following identities as postulates. $\cos(0°)=1$; $\sin(90°)=1$; $\cos(180°)=-1$. Do not make use of any trigonometry identity that you haven't proved previously in this development, or haven't been told you may assume as a postulate.

a) a) Give the triangle proof of the Pythagorean identity $\cos^2(o)+\sin^2(o)=1$; b) Give the circle proof of the Pythagorean identity.

b) Make use of the postulates I2) $\cos(0°)=1$, I3) $\sin(90°)=1$, I4) $\cos(180°)=-1$, to prove each of the following identities. I5) $\sin(0°)=0$; I6) $\cos(90°)=0$; I7) $\sin(180°)=0$.

c) Make use of Analytic Geometry to derive the cosine subtraction formula ... $\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$.

d) Make use of the cosine subtraction formula to prove that cos is an even function, i.e. prove $\cos(-o)= -\cos(o)$.

e) Prove $\cos(90°-o)=\sin(o)$ and $\cos(o-180°)= -\cos(o)$, then make use of these identities to prove that sin is an odd function, i.e. prove $\sin(-o)= -\sin(o)$.

f) Make use of the cosine subtraction formula to derive the cosine addition formula, i.e. derive $\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$

g) Make use of the cosine addition formula to derive the sine addition formula i.e. derive $\sin(a+b)= \cos(a)\sin(b)+\sin(a)\cos(b)$.

h) Make use of the sine addition formula to derive the sine subtraction formula i.e.
derive $\sin(a-b)= \cos(a)\sin(b)-\sin(a)\cos(b)$

i) Make use of the cosine and sine addition formulas to derive the tangent addition formula i.e. derive
$\tan(a+b)= \{\tan(a)+\tan(b)\}/\{1-\tan(a)\tan(b)\}$

j) Prove tangent is an odd function, i.e. prove $\tan(-o)= -\tan(o)$.

k) Make use of the tangent addition formula to derive the tangent subtraction formula i.e. derive
$\tan(a-b)= \{\tan(a)-\tan(b)\}/\{1+\tan(a)\tan(b)\}$

l) Make use of the cosine addition formula to derive the first cosine double angle formulas, then make use of first cosine double angle formula to derive the 2nd and 3rd double angle formulas.
I) $\cos(2o)= \cos^2(o)-\sin^2(o)$
II) $\cos(2o)= 2\cos^2(o)-1$
III) $\cos(2o)= 1-\sin^2(o)$

m) Make use of the cosine double angle formula(s) to derive the cosine and sine squared formulas.
I) $\cos^2(o)= \{1+\cos(2o)\}/2$
II) $\sin^2(o)= \{1-\cos(2o)\}/2$

n) Make use of the cosine and sine squared angle formulas to derive the cosine, sine and tangent half angle formulas.

```
      (o)          {1+cos(o)}
I) cos(-)= +/- sqrt{--------}
      (2)          {   2    }
```

```
       (o)          {1-cos(o)}
II) sin(-)= +/- sqrt{--------}
       (2)          {   2    }
```

```
        (o)          {1-cos(o)}
III) tan(-)= +/- sqrt{--------}
        (2)          {1+cos(o)}
```

------- End of Optional Problem Set (Regular Option) --------

You may want to prepare for doing the (honors) option by studying or doing all or part of the regular option.

3 Honors Option)

Prove the Pythagorean identity. Make use of coordinate geometry to prove the cosine subtraction formula. Now assume the following as postulates, I2) cos(0°)=1 : I3) sin(90°)=1 : I4) cos(180°)=-1. From there go on to derive the tangent addition and subtraction formulas, and then the tangent half angle formula. Do not refer to a list of identities that gives you the order in which identities are to be derived. You are to figure this out for yourself. Do not make use of any trigonometry identity that you haven't yet proved, or that you haven't been told you may assume as a postulate.

A Mistake to Avoid When Doing Proofs

When asked to prove an identity

a(o)=b(o)

a common (but in general not valid) way to do this is to manipulate both sides of the identity one or more times until it takes a form that is known to be true. This technique is illustrated below.

(Remember -> means implies)

```
a(o)=b(o) ->     <- identity you are trying to prove
c(o)=d(o) ->
e(o)=f(o) ->
y(o)=z(o)        <- this identity is known to be true
```

In general, a proof of this form does not prove a(o)=b(o). What a proof of this form does do is show that if a(o)=b(o) is true, then y(o)=z(o) is true. In other words this proof shows that a(o)=b(o) -> y(o)=z(o), i.e. a(o)=b(o) implies y(o)=z(o).

If A implies B, B does not necessarily imply A. For example,

R is an even number -> R is an integer .. true
R is an integer -> R is an even number .. not true

It is for this reason that this proof does not (in general) prove that a(o)=b(o) is true.

A proof of this form is valid (in general) if each of the steps are reversible.

if

a(o)=b(o) ->
c(o)=d(o)

and

c(o)=d(o) ->
a(o)=b(o)

Then

a(o)=b(o) ->
c(o)=d(o)

is a reversible step.

If each of the steps in the preceding proof are reversible, then this proof can be reversed and re-written as

y(o)=z(o) ->   <- this identity is known to be true
e(o)=f(o) ->
c(o)=d(o) ->
a(o)=b(o)   <- This identity is now proven

This is a valid proof of a(o)=b(o).

For a concrete example of this type of proof, see proof of problem 4.5.4, in the 'Law of Cosines, Law of Sines' section.

4) The following identities come in handy, learn to recognize them.
Prove: a) cos^2(o)-sin^2(o)=cos(2o); b) sin(o)cos(o)=sin(2o)/2

5) These two identities are important and should be learned. a) Make use of the cosine subtraction formula to prove cos(90°-o)=sin(o); b) Make use of the sine subtraction formula to prove sin(90°-o)=cos(o)

6) The following identities come in handy, learn them.
a) Prove: cos(o+180°)= -cos(o); cos(o-180°)= -cos(o)
b) Prove: sin(o+180°)= -sin(o); sin(o-180°)= -sin(o)
c) Prove: tan(o+180°)= tan(o); tan(o-180°)= tan(o)
d) Prove these identities geometrically.

7.7) a) Make use of sin(90°-o)=cos(o) to prove cos(90°-o)=sin(o)
b) Make use of cos(90°-o)=sin(o) to prove sin(90°-o)=cos(o)

8) a) Prove: tan(90°-a)=cot(a); b) cot(90°-a)=tan(a)

9) a) Make use of cos(o-90°)=sin(o) to prove sin(o+90°)=cos(o)
b) Make use of sin(o-90°)=-cos(o) to prove sin(o+90°)=cos(o)

10) a) Prove: tan(a)+tan(b)={1-tan(a)tan(b)}tan(a+b);
b) Make use of the previous identity (part a) to derive a similar expression equal to tan(a)-tan(b).

11.7) a) Prove: a) 1+tan^2(o)=sec(o); b) 1+cot^2(o)=csc^2(o)

12) a) Assuming tan(o)=sqrt(3), what is tan(o/2)? b) Make use of a 30°-60°-90° triangle to verify your answer is correct. c) Assuming tan(o)=a, (0°<o<90°), what is tan(a/2)?

13) Derive the cotangent addition and subtraction formulas, in terms of cotangent, using the following two methods. a) Use a method similar to the method this book used to derive the tangent addition formula. b) Make use of the tangent addition and subtraction formulas. In doing this, do not make use of cos or sin functions.

14) a) Derive 'the' cos(a+b+c)= ... formula.
b) Derive 'the' sin(a+b+c)= ... formula.
c) Derive 'the' tan(a+b+c)= ... formula.

15) Derive the following secant and cosecant addition formulas.
a) sec(a+b)= sec(a)csc(a)sec(b)csc(b)/{csc(a)csc(b)+sec(a)sec(b)}
b) csc(a+b)= sec(a)csc(a)sec(b)csc(b)/{sec(a)csc(b)+csc(a)sec(b)}

```
          sin^2(o)
16) Prove ---------- = tan^2(o)
         1-sin^2(o)
```

17) Prove $$\tan(o+45°)=\frac{1+\tan(o)}{1-\tan(o)}$$

18.7) Prove $$\frac{\sin(a-b)}{\sin(a+b)}=\frac{\cot(b)-\cot(a)}{\cot(b)+\cot(a)}$$

19) Prove $$\cot(a)+\cot(b)=\frac{\sin(a+b)}{\sin(a)\sin(b)}$$

20.7) Prove $$\frac{1-\tan(o)}{1+\tan(o)}=\frac{\cot(o)-1}{\cot(o)+1}$$

21) Prove $\cos^4(o)-\sin^4(o) = \cos^2(o)-\sin^2(o)$
(Hint if needed, see proof of next problem)

22.7) Prove $\cos^4(o)-\sin^4(o) = 1-2\sin^2(o)$

23) Prove $$\frac{1-\tan^2(o)}{1-\cot^2(o)} = 1-\sec^2(o)$$

24.7) Prove $$\frac{\tan(a)+\tan(b)}{\tan(a)\tan(b)} = \cot(a)+\cot(b)$$

25.7) Doing this problem will help to prepare you for the problems that follow. .. $\tan(x)=a$: a) $\cos(x)=?$; b) $\sin(x)=?$

26) A ray with an orientation of 'a' has a slope of 'm'. A ray with an orientation of 'b' has a slope of 'n'. What is the slope of a ray with an orientation of a+b?

27) A ray (in the 1st quadrant) with an orientation of o has a slope of 2. What is the slope of a ray with an orientation of o/2?

In the following four problems, do not make use of inverse trigonometry functions unless you are asked to do so.

28) Two rays have their end point on the lower left corner of a square. One of these rays passes through the midpoint of the top side of the square, the other ray passes through the midpoint of the right side of the square. a) Determine the sine of the acute angle defined by these rays. b) Make use of the inverse sine function on your calculator to find this angle in degrees.

29) Ray 1 has a slope of 1, ray 2 has a slope of 2. Both rays have their end point at the origin and are in the first quadrant. In an alternate (rotated) coordinate system where both ray's end point are still at the origin, and ray 1 lies along the positive x axis, what is the slope of ray 2?

30) Prove Angles Are Equal
The following rays have their end point at the origin and exist in the 1st quadrant. Ray A has a slope of 3, ray B has a slope of 1, ray C has a slope of 2 and ray D has a slope of 3/4. Prove that the angles formed by the ray A and ray B equals the angle formed by ray C and ray D.

31) Find the slope of the angle bisector of the angle that is formed by the following two rays. Both rays have a common end point at the origin (0,0) and exist in the in the 1st quadrant. One ray has a slope of 3, the other ray has a slope of 5.

32) Where $a+b+c+d=180°$,
prove $\sin(a)\sin(b)+\sin(c)\sin(d)=\sin(a+c)\sin(b+c)$

33) Where $a+b+c=90°$ and a,b,c are all positive angles, prove
$\tan(a)\tan(b)+\tan(b)\tan(c)+\tan(c)\tan(a)=1$

^^^^^^^^^

8 Periodic Events and Functions
  Distance Rate & Time
  Trigonometry Graphing Introduction

^^^^^^^^^

D=R*T is the governing equation of DRT (Distance Rate Time) problems. In several DRT problems below, the word problem gives two of these parameters (variables). Once you determine these two, solve for the remaining unknown parameter.

Example

A hose can fill a pool in 7 hours. If the hose fills the pool at this rate for two hours, how full will the pool be? D

A hose can fill a pool in 7 hours -> R = 1 pool / 7 hours
If the hose fills the pool at this rate for 2 hours -> T = 2 hours

We have R=rate and T=time, So we need to solve for D=distance

```
       1 Pool              2
D=R*T= ------- * 2 hours = - Pool, so the pool is 2/7 full.
       7 hours             7
```

<u>Distance Rate Time Problem Set</u>

1) A train travels 78 miles per hour for 3 hours. How far does the travel? D

2) If a bug goes around a light 3 times every 2 seconds. At what rate does the fly rotate around the light? R

3) An airplane averages 520 miles per hour on a trip of 1320 miles. How long does the trip take? T

4) If a hose is able to fill a pool to 4/5 capacity in 3 hours, at what rate does the hose fill the pool? R

5) A car is to make a trip to Salt Lake City. If it goes 5/8 of the way in 2 1/3 hours. What is the rate at which this car travels towards Salt Lake City? R

6) If a hose can fill a pool in 13 hours. How long does it take to fill the pool to 2/3 capacity? T

7) If a hiker can climb 2/7 of the way up a mountain in 3 hours. How long will it take to hiker to climb 4/5 of the way? T

8) Children can walk to school in 5/13 of an hour. If they walk towards school at this rate for 2/7 of an hour, what proportion of the way will they walk towards school? D

9) A satellite orbits the earth in 3/2 of an hour. In 5 3/8 hours how many times does it go around the earth? D

10) A hose fills a pool in 15 hours. A leak flowing at a constant rate in the bottom of the pool can empty the pool in 8 days. a) At what rate does the hose fill the pool? b) At what rate does the leak empty the pool? c) At what rate do the hose and the leak acting together fill the pool? d) Assuming the pool is empty, how long will it take to fill the pool? T

11.8) One hose can fill a swimming pool in 4 hours. Another hose can fill a pool in 3 hours. How long will it take for both of these hoses working together to fill the pool? T

12) An escalator can move a person from the bottom to the top or from the top to the bottom in 1 minute. If the escalator isn't moving, a person can climb up the stairs in 25 seconds. a) If the escalator is moving upward, how long would it take the person to climb to the top? b) If the escalator is moving downward, how long would it take for the person to climb to the top? c) The speed of the escalator is changed. At this new speed it takes the man 20 seconds to climb to the top. How fast is the escalator moving?

13) Thieves robbed a bank and made their getaway down the highway at a speed of 80 miles per hour. 20 minutes later the police started chasing the robbers traveling at a speed of 105 miles per hour. a) How long will it take for the police to catch the robbers? b) How far will the police have to travel to catch the robbers? D

Periodic Events

1) A satellite orbits a planet in its equatorial plane once every 11 hours. The planet rotates about its axis (in the same direction) once every 5 hours. a) At what rate does the satellite orbit the planet? b) At what rate does the planet revolve about its axis? c) From the perspective of a person on the planets equator, at what rate (how often) does the satellite pass overhead? d) Do again assuming the orbit of the satellite and the revolution of the planet are in opposite directions.

2) One of Jupiter's moon's revolves around the planet in its equatorial plane once every 42 hr 30 min. Jupiter rotates about its axis (in the same direction) once every 9 hours 55 minutes. a) What is the length of time between consecutive passages of the moon over the same point on Jupiter's equator? b) Do again assume Jupiter and its moon revolve/orbit in opposite directions.

Period and Frequency of Periodic Events

If an event happens on a recurring basis, such that the timing between events is constant, such an event is called a periodic event. Examples of a periodic events are a persons birthday, and the ticking of a clock.

Speaking loosely, the period of a periodic event is the time between event completions (or starts) and the frequency is of a periodic event is how often the events are happening. Consider a rotating top where one rotation of the top is considered to be the event. If the top is spinning 25 times every second, the period of this event is 1/25 second. The frequency of this event is 25/second. (Often referred to as 25 rotations/second). Period and frequency are reciprocals of each other, i.e. frequency = 1/period and period = 1/frequency.

Exercise: Electricity delivered to homes and businesses does not flow in one direction but flows back and forth in a wire. In the United States this flowing back and forth occurs 60 times each second. Taking one cycle back and forth as one event, a) What is the period?; b) What is the frequency? c) What is the relationship between period and frequency?

Useful Conversions
1 inch = 2.54 centimeters
1 mile = 5280 feet
1 hour = 3600 seconds

3) The diameter of a motorcycle's front wheel is 79 centimeters. The diameter a motorcycle's rear wheel is 80 centimeters. If the motorcycle is traveling at 65 miles per hour. a) What is the period of rotation of each of the tires? b) What is the frequency of rotation of each of the tires? c) How often are the valve stems (air intakes) of both tires in the same angular position? d) What would the diameter the front wheel need to be in order that both valve stems be at the same angular position every 10 seconds? e) Is there more than one correct answer for question in 'd'? If so, give another answer.

Periodic Functions (cosine and sine)

R is a ray with its end point at the origin. Its orientation is x, meaning that it intersects a unit circle C centered at the origin at {cos(x), sin(x)}. If R has an initial orientation of x=Xo (a constant), and x increases from there, R will be spinning in a counter clockwise direction. (Take a few moments to verify this for yourself). Not until x = Xo + 1 revolution, or o = Xo + 360 degrees, or x= Xo + 2Pi radians will the ray return to its original orientation. It seems reasonable to guess therefore that the graph of the functions y=cos(x) and y=sin(x) would repeat once every revolution or 360 degrees or 2 Pi radians as x increases (and not more often than that). If we graph y=cos(x) or y=sin(x) using a graphing calculator or computer we can see this is true.

Exercise: Using a graphing calculator or a computer,

a) Graph y=cos(x) from -360° to 720° (720°=2*360°)

b) Graph y=sin(x) from -360° to 720°

How many times does your graph of y=cos(x) and y=sin(x) repeat? If you said 3 times you are correct. Speaking loosely: 1. A function that repeats over its entire domain is a periodic function and 2. If a periodic function repeats once on the domain from x=a to x=a+b, then b is the wavelength of that function. cos(x) and sin(x) each have a wavelength of 2Pi radians. More formally, a function is a periodic function with a wavelength of L, if f(x)=f(x+L) (L>0) (-inf < x < inf), and L is the least positive number for which this is true. Take a few moments to assure yourself this formal definition of a periodic function and wavelength makes sense and is true, i.e. is it in agreement with the less formal definitions?

Wavelength

In the following discussion cos is used, but the discussion also applies to sin. Also radians are used, but the discussion could be adapted to apply to degrees or revolutions.

cos has a wavelength of 2Pi rads. The graph of y=cos(x) from x = Xo to Xo + 2Pi rads is one wavelength of the graph y=cos(x). To figure out the wavelength of y=cos(x), y=cos(ax), y=cos(x/a) calculate how much the x needs to change, in order for what is inside the parenthesis to change by 2Pi rads. This amount is the wavelength of the function. Take a few moments to understand for yourself why this method of determining the wavelength of a cosine function is valid.

Example: Calculate the wavelength of y=cos(ax) ...
For ax to change from 0 to 2Pi, x must change from 0 to 2Pi/a.

therefore the wavelength of y=cos(ax) is x=2Pi/a

Exercise: Calculate the wavelength of y=cos(x/u).

Exercise: What is the wavelength of y=sin(3x) in degrees?

Exercise: What is the wavelength of y=cos(x/2) in degrees?

If you haven't already noticed, the distance from crest to crest (or from trough to trough) of a cosine function (or sine function) is the wavelength of the function. Also the distance from a zero crossing where the function is increasing (or decreasing), to the next zero crossing where the function is increasing (or decreasing) is the wavelength of the function.

Period

In the discussion of periodic functions and wavelength until now, the domain variable has been x. By convention, a domain variable x implies distance. Two points on the x axis are thought of as being separated by some distance. An other common domain type for cosine or sine functions is time. By convention, if time is the domain type of a function, the domain variable is t. Two points on the t axis are though of as being separated in time. y=cos(t) is an example of a cosine function where the domain type is time.

Where time, not distance is the domain of a periodic function, the word wavelength isn't used, instead the word period is used. If the formal definition of wavelength of a function is slightly modified, it applies to period of a function. period P (P>0) of a time domain periodic function f is the least amount of time t can change so that (in general) f(t)=f(t+P). The situation is quite similar to where the domain is distance. The period of y=cos(t) or of y=sin(t) is 2Pi. If seconds is the unit of time used, then the period of y=sin(t) is 2Pi seconds. When time is the domain, radian measure needs to be used.

Example: What is the period of y=sin(x/3)?

> (1/3)x=2Pi -> x=2Pi/(1/3) -> x=6Pi -> period=6Pi or approximately 18.85.

Frequency

Frequency (or F) is how often something happens. The frequency of the sun rising is once a day or (1 rise)/day or just 1/day. The electricity that comes into our homes cycles back and forth 60 times a second, its frequency is 60 cycles per second or 60 cycles/second or just 60/sec. Frequency (or F) and period (or P), are reciprocals of each other, i.e. F=1/P and P=1/F. Given that the frequency of the electricity that comes into our homes is 60/sec, the period of this electricity is 1/(60/sec) or 1/60 second.

Example: What is the frequency of y=cos(ax)?

ax=2Pi -> x=2Pi/a -> period = 2Pi/a -> frequency = 1/(2Pi/a) -> frequency = a/2Pi

Periodic Functions Problem Set

1) R is a ray with end point at the origin. The orientation of R is t (time). C is the unit circle centered at the origin. P is the point of intersection of R and C. The position of P is represented by the parametric point {cos(t),sin(t)}. a) As t increases from 0 to 2Pi, how far does P move? b) How much time passes as t increases from 0 to 2Pi?. c) How fast (on average) does the point {cos(t),sin(t)} move going once around the circle? d) What is the period of the point {cos(t),sin(t)} moving around the circle C? {compare this with the periods of cos(t) and sin(t)}.

2) Calculate and then give the wavelength or period of the following functions. Where appropriate, also calculate the frequencies of these functions. Give all numerical answers using at least four digits of precision.
a) y=sin(x); b) y=cos(x); c) y=sin(t)
d) y=cos(8t); e) y=cos(t/2); f) y=sin(12x)
g) y=cos(a*x); h) y=sin(t/a); i) y=sin(x/a)

3) I) Give a cosine function whose wavelength is each of the following. II) Give a sine function whose period is each of the following. a) 2Pi; b) 1/3; c) Pi/5; d) 1; e) 7; f) U

4 Calculate the frequency of the following. a) cos(t); b) sin(10Pi*t); c) cos(t/3); d) sin(t/Pi); e) cos(at).

5) Calculate and then give a function (choose cos or sin) whose whose frequency is each of the following. a) Pi; b) 2/Pi; d) 1; e) 60; f) V

If sound exists at a given location in air, then the air pressure at this location is changing. This rapid change of air pressure is the sound. A sound source sets up pressure waves in the air that travel away from the source at the speed of sound, which is about 343 meters per second. If for example the frequency of a sound is 500 Hz, then the wavelength of the sound wave traveling through the air due to this source would be
(343 meters/second)/(500/second) = 343/500 meters
Take a few moments to assure yourself this is true. The following equation relates wavelength, speed and frequency; of sound.

speed (of traveling wave) = wavelength * frequency

Take a few moments to assure yourself this equation is true.

6) Middle C on the piano has a frequency of 440 Hz. Assuming the speed of sound is 343 meters per second, what is the wavelength of a middle C sound wave that travels through the air?

Bonus Problems ... The Doppler Effect

1a) i) A car approaches a stationary fire truck at 65 mph. The fire truck siren emits a sound of 1000 Hz. Assume sound travels at a speed of 1125 ft/sec. What is the frequency of the sound that the driver of the car hears from the siren? ii) In problem i, if the car is moving away from the siren at 65 mph, what is the frequency of the sound that the driver hears from the siren?

1b) Derive a Doppler equation that gives the frequency of sound heard by an observer when the observer is moving towards (or away from) the sound source.

1c) Make use of the equation you derived in problem 1b to solve the problems given in 1a. Did you get the same answers?

2) a) Derive a Doppler equation that gives the frequency of sound heard by an observer when the sound source is moving towards (or away from) the observer. ... Make use of this equation to solve the following problems. b) A fire truck siren emits a 1000 Hz sound. The fire truck is moving towards a stationary car at a speed of 75 mph. Assuming sound travels at 1125 ft/sec, what is the frequency that the occupants of the car hear from the siren? c) Assume in part 'b' that the fire truck is moving away from (not towards) the car. What frequency do the occupants of the car hear from the siren?

3) a) Derive a Doppler equation that gives the frequency of sound heard from a sound source when the sound source is moving towards (or away from) an observer AND the observer is moving towards (or away from) the sound source. Make use of this equation to solve the following problems. ... A car and a fire truck are traveling in opposite directions on a straight road. The siren of the fire truck emits a sound of 1000 Hz. The fire truck travels at 75 mph, the car at 65 mph. Assuming sound travels at 1125 ft/sec, what is the frequency of the sound heard by the driver as b) the fire truck and the car are approaching each other? c) after the car and the fire truck have passed each other?

4) The speed of sound in the air is a function of the temperature of the air. You wish to determine the speed of sound by measuring the difference of the frequency of a sound source, and the perceived frequency of this sound source when you are moving towards or away from it, at a known speed. Derive an equation which will allow you to do this.

^^^^^^^^^^^^^^^^^^^^^^^^^

9 Inverse Trigonometry Functions

^^^^^^^^^^^^^^^^^^^^^^^^^

Functions, Inverse Functions
An Important Theorem Relating the Two

Before discussing inverse trigonometry functions specifically, we will discuss functions and inverse functions in general. To begin with, an inverse function is a function. There is more than one way to describe what a function is, we take the approach here that is most suited to our needs. A function provides one output in response to an input or combination of inputs. Here we deal only with functions that accept only one input at a time. This is known as a single variable function. The set of allowable inputs of a function is called the domain of the function. The set of possible outputs of a function is called the range of the function. Whenever one domain element (which in our case is a single number, angle or variable) is inputted into a function, the function in accordance with a given rule will provide one output. For a given input, the output of a particular function will always be the same. If an input is provided to a function that is not a domain element, the function will not provide an output.

Consider the function F where x -> F -> 3x. What this is saying is that if x is inputted into the function F, F will provide 3x as an output. In more standard notation this function is expressed as y=3x. Here is another way to view the situation. Consider the function x -> F -> y. Here we see a variable x passing through F and becoming the variable y. Consider x -> F -> xf, here the variable x passes through the function F, which "changes it" to xf, some other variable. Our placing the small f, after the x indicates that this variable is equal to the x, after it has passed through the function (big letter) F. We will use this variable naming technique again.

The defining characteristic of an inverse function or inverse, is that it undoes what a function does. For example, if the function F, upon accepting x as its input, outputs the value xf, then an inverse function of F, (which we refer to as F') upon accepting xf as its input will output the value of xff' or x. In our alternate functional notation this process is illustrated by the following expression.

x -> F -> xf -> F' -> xff' = x

In standard notation this is expressed as
F'{F(x)}=x <--- This implies F' is the inverse function of F
Loosely speaking, F converts a value x into a value xf, but F' which is the inverse function of F, converts xf back into x. F' undoes what F does. Not all functions have inverses.

Q1: What is an inverse function? What does an inverse function do?

Theorem 1.9 If F' is an inverse of F, then F is also an inverse of F'.. only if, the only inputs F' will accept, are outputs of F.

Proof:

F' being an inverse function of F implies the following expression.

i) x -> F -> xf -> F' -> xff' = x

This expression can be re-written as

ii) xf -> F' -> xff' = x -> F -> xf

Refer to i) x is a variable representing any or all of the domain elements (allowable inputs) of F, therefore xf is representative of all range elements (possible outputs) of F. xf passes through F', an inverse of F, and in doing this xf becomes xff'=x.

Refer to ii) xf passes through F', becoming x, x then passes through F becoming xf. We see that if the variable xf passes through F' and then through F, it remains unchanged. Therefore F acts as the inverse function of F' for any variable of the form xf, which is all possible outputs (range elements) of F. Therefore so long as inputs to F' are outputs of F, F acts as the inverse of F'.

(Refer to the next line in the paragraph below)

y (which is not an output of F) -> F' -> yf' -> F -> yf'f != y

What happens if F' can accept an input y which is not a possible output of F? If yf' can not pass through F, F can not be an inverse of F'. If yf' can pass through F, yf' becomes yf'f. Since y is not a possible output of F, yf'f != y, therefore in this instance F is not acting as an inverse of F' either. Therefore if F' accepts input(s) which are not outputs of F, F is not an inverse of F'.

Proof Complete

Theorem: 2.9 If F and G are functions and G is an inverse of F. Then F and G are inverses of each other if and only if, the domain of each function is equal to the range of the other function.

The proof of this theorem is left to the student.

There are two important take aways from the preceding theorems.

I) If F is a function and F' is an inverse of F, there is a (good) possibility that F and F' are inverses of each other. Whether or not this is true can be easily determined by looking and the domain and range (possible inputs and outputs of each function).

II) If F is a function, and F' is an inverse of F, but F is not an inverse of F'. F' can be modified so that F will be its inverse. This is done by restricting the domain of F' so that it accepts only outputs of F. o=arccos(x) for example, is defined as a particular angle such that cos(o)=x. Therefore cos is an inverse of arccos. However because cos does accept inputs which are not outputs of arccos, arccos is not an inverse of cos. (see theorem 1.9). A domain restricted form of cos, i.e. Cos has been defined such that the only inputs Cos will accept are outputs of arccos, therefore invcos is an inverse of Cos. arccos and Cos are inverses of each other. arccos and cos are not inverses of each other.

The trigonometry functions cos, sin and tan do not have inverses. We will see later in this section if their domains are restricted, they become different but similar functions that do have inverses.

Inverse Functions of the 6 Trigonometry Functions

In the following discussion, cos and arccos and invcos are used as example trigonometry and inverse trigonometry functions. However the discussion also applies to all of the trigonometry and inverse trigonometry functions, i.e. sine, cosine, tangent, cosecant, secant and cotangent, and their 'inverses'.

If we know the value of an angle o, and desire to know what cos(o) is, we can use the cosine function of a calculator to find the answer. However if we are told that cosine of an angle is 1/2, can a calculator be used to determine what THE angle is? If cos(o)=1/2 can we determine o? The answer is no because there is more than one angle whose cosine is 1/2. Cosine of 60 degrees equals 1/2, however cosine is an even function so cosine of -60 degrees is also equal to 1/2. Keeping this and the circle definition of cosine in mind, it should be evident that
cos(60° + 360°*n) and cos(-60° + 360°*n) n=(..-3,-2,-1,0,1,2,3..) are also equal to 1/2. There are infinitely many angles whose cosine is any particular (allowed) value, such as 1/2. Using the arccos (or invcos) function of a calculator we can find one of these angles. For example if you ensure a calculator is in degrees mode and then enter 0.5 into the calculator, then press the arccos key, the calculator will return an angle whose cosine is 1/2, i.e. the calculator will return 60 (degrees) = arccos(1/2). We see cos{arccos(1/2)}=1/2. In general cos{arccos(x)}=x. {By definition arccos(x) is an angle (returned by the calculator) whose cosine is x}. Therefore cos is an inverse function of arccos. We desire to find an inverse of cosine, but this is seemingly impossible.

For example ... the following is true
+60° -> cos -> 1/2 -> arccos -> +60°, the following is also true.
-60° -> cos -> 1/2 -> arccos -> +60°. Therefore any function that would act as an inverse of cosine if the argument (the independent variable of a function) is 60°, will not act as an inverse cosine function for cosine when the argument is -60 degrees.

Q2: Explain why cos does not have an inverse and why a domain restriction of cos could (possibly) 'fix' this situation.

The preceding reasoning illustrates the fact that cosine does not have an inverse function. Yet many calculators have what they refer to as inverse trigonometry functions. These are labeled as arccos or invcos. How is this justified? The function the calculator refers to as arccos is an inverse of a domain restricted cosine function. This domain restricted cosine function does not for example accept both both 60 degrees and -60 degrees as inputs. (If so, arccos could not be its inverse).

We desire that arccos and a domain restricted cosine function be inverses of each other if possible. Any domain restriction on a trigonometry function needs to be done in such a way that the range of the trigonometry function is not diminished. (For cosine a domain must be chosen so that the range continues to include all numbers from -1 to 1]. We choose the following set of domain restrictions because they match the ranges of the corresponding inverse trigonometry functions used on calculators. Given that cos is an inverse of arccos, this matching is important if there is to be any hope of arccos and the domain restricted cosine function being inverses of each other, (see theorem 2.9)

cosine and secant .. domain restricted from 0 to 180 degrees.
sine and cosecant .. domain restricted from -90 to 90 degrees.
tangent and cotangent .. domain restricted from -90 to 90 degrees.

We need to differentiate trigonometry functions that are domain restricted and the ones which are not. The following are the domain restricted trigonometry functions. Cos, Sin, Tan, Sec, Csc, Cot. (Notice the capitalized first letters). Inverses of these domain restricted trig functions are arccos, arcsin, arctan, arcsec, arccsc and arccot respectively. Other names commonly used for these inverse trig functions are invcos, invsin, invtan, invsec, invcsc and invcot respectively.

As stated previously, cos is an inverse of invcos. Since [by design] Cos can accept anything as an input that is outputted by arccos, Cos is also an inverse of arccos. Cos{arccos(x)}=x

The domain and range of Cos and arccos are illustrated below.

(0° to 180°) -> Cos -> (1 to -1)
(1 to -1) -> arccos -> (0° to 180°)

Given that Cos is an inverse of arccos, since the only inputs Cos will accept are outputs of arccos, arccos is also an inverse of Cos. (see theorem 1.9)

Given that Cos is an inverse of arccos, since the domain of each of these functions is equal to the range the other, Cos and arccos are inverses of each other. (see theorem 2.9) (Theorem 2.9 is a replacement for theorem 1.9, but is easier to use and to remember).

Q3: a) Give an inverse for the following functions (if any).
cos(o), sin(o), tan(o), sec(o), csc(o), cot(o)
Cos(o), Sin(o), Tan(o), Sec(o), Csc(o), Cot(o)
arccos(x), arcsin(x), arctan(x), arcsec(x), arccsc(x), arccot(x)

Q4: What are the domain and range of a) Each of the trig functions? b) Domain restricted trig functions? c) Each of the inverse trig functions?

Each of the following statements are true.

Cos{arccos(x)}=x; Sin{arcsin(x)}=x; Tan{arctan(x)}=x
Sec{arcsec(x)}=x; Csc{arccsc(x)}=x; Cot{arccot(x)}=x

arccos{Cos(o)}=o, arcsin{Sin(o)}=o, arctan{Tan(o)}=o
arcsec{Sec(o)}=o, arccsc{Csc(o)}=o, arccot{Cot(o)}=o

cos{arccos(x)}=x; sin{arcsin(x)}=x; tan{arctan(x)}=x
sec{arcsec(x)}=x; csc{arccsc(x)}=x; cot{arccot(x)}=x

The following trigonometry statements are not universally true.
(the domain unrestricted trigonometry functions do not have inverses)

arccos{cos(o)}=o, arcsin{sin(o)}=o, arctan{tan(o)}=o
arcsec{sec(o)}=o, arccsc{csc(o)}=o, arccot{cot(o)}=o

Whenever you see the function invcos(x), or arccos(x) it is useful to mentally replace this function with the words ... "An angle (returned by the calculator) whose cosine is x". Likewise for any of the other inverse trig functions. Experience has shown that if these words are kept in mind when solving problems involving invcos(x) or arccos(x), solving the problems will be easier.

Inverse Functions Problem Set

1) If F{G(x)} equals x, then ____ is the inverse of ____.

2) Find the value of the following (in degrees).
a) arccos(0.5); b) invsin(0.32); c) invtan(1), d) arctan(-1);
e) invsec(4); f) arccsc(-6); g) invcot(11); h) arccot(0.1)

3) Determine the value of each of the following expressions. Use a calculator to check your answers. a) cos{invcos(0.7)};
b) sin{invsin(-0.21)}; c) tan{invtan(3)}

4) Determine the value of each of the following expressions. Use a calculator to check your answers. a) invcos{cos(32°)};
b) arccos{cos(400°)}; c) arctan{tan(-10°)};
d) invtan{tan(350°)}

5.9) Determine all angles 'o' such that
a) cos(o)=0.21; b) sin(o)=-0.77; c) tan(o)=-1
Determine all angles from -360° to 720° such that
a') cos(o)=0.21; b') sin(o)=-0.77; c') tan(o)=-1

6) Which of the following is true?
a) cos{invcos(x)}=x; b) invcos{cos(o)}=o; c) Cos{invcos(x)}=x
d) sin{invcos(x)}=x; e) invsin{sin(o)}=o; f) Sin{invcos(x)}=x
g) tan{invcos(x)}=x; h) invtan{tan(o)}=o; i) Tan{invcos(x)}=x

7.9) Simplify the following expressions, the simplified forms will not contain any trigonometry functions. Hint: Upon seeing an inverse function, invcos(t) for example, remember to think the words. The angle (returned by the calculator) whose cosine is t. Its best to leave the words (returned by the calculator) out.

a) cos{arccos(x)}; b) cos{arcsin(x)}; c) cos{arctan(x)};
d) cos{arcsec(x)}; e) cos{arccsc(x)}; f) cos{arccot(x)};

g) sin{invcos(x)}; h) sin{invsin(x)}; i) sin{invtan(x)};
j) sin{invsec(x)}; k) sin(invcsc(x)}; l) sin(invcot(x)};

m) tan{arccos(x)}; n) tan{arcsin(x)}; o) tan{arctan(x)};
p) tan{arcsec(x)}; q) tan{arccsc(x)}; r) tan{arccot(x)};

s) sec{invsin(x)}; t) sec{invcos(x)}; u) sec{invtan(x)};
v) sec{invsec(x)}; w) sec{invcsc(x)}; x) sec{invcot(x)};

y) csc{arcsin(x)}; z) csc{arccos(x)}; aa) csc{arctan(x)};
ab) csc{arcsec(x)}; ac) csc{arccsc(x)}; ad) csc(arccot(x)};

ae) cot{invsin(x)}; af) cot{invcos(x)}; ag) cot{invtan(x)};
ah) cot{invcsc(x)}; ai) cot{invsec(x)}; aj) cot{invcot(x)}

8) Prove: tan{invtan(a)+invtan(b)} = (a+b)/(1-a*b)

9) Prove: sin{arccos(a)+arccos(b)} = a*sqrt(1-b^2)+b*sqrt(1-a^2)

10) Prove: cos{arccos(a)+arcsin(b)} = a*sqrt(1-b^2)-b*sqrt(1-a^2)

11) sec{arctan(a)+arccsc(b)}= ? .. express '?' algebraically

12.9) Assuming y=f(x) is a function that has an inverse, prove that the function obtained by exchanging the x and the y {i.e. x=f(y)} is the inverse of y=f(x).

13) I) Make use of the fact that an inverse of y=f(x) (if one exists) can be obtained by switching the x and y, to determine an inverse of each of the following functions. II) Then check your answers by substituting each function into its inverse function and simplifying. a) y=3-2x; b) y=3+x; c) y=1-x.

14.9) Make use of the fact that f{g(x)}=x, implies f is an inverse function of g to calculate an inverse function of each of the following functions. I) These are the same problems as were given in the previous problem. If you did both sets of problems correctly you should get the same answers both times.
a) y=3-2x; b) y=3+x; c) y=1-x.

15) a) Prove: If F' is an inverse of F, then F is also an inverse of F' .. only if, the only inputs F' will accept, are outputs of F.

16) a) Prove: If F and G are functions and G is an inverse of F. Then F and G are inverses of each other if and only if, the domain of each function is equal to the range of the other function.
b) Go through steps to calculate an inverse of y=mx+b. Verify that the (potential) inverse you calculated is an inverse. c) Make use of this theorem to see if y=mx+b and its inverse are inverses of each other.

17) Make a table of the domain and range of each the following. Cos, Sin, Tan, arccos, arcsin, arctan, Make use of this information to help prove that Cos, Sin, Tan and their inverses are inverses of each other.

<u>Determining the Inverse of Composite Functions</u> (Method 1)

Assuming y=Cos(3x+7) has an inverse. Determine 'it' by making use of the fact that the inverse of y=f(x), is x=f(y). Once you do this, verify the result you got is an inverse of y=Cos(3x+7)

Try to do this problem on your own before looking at the solution that follows. STOP

An inverse of $y=\cos(3x+7)$ is

$x=\cos(3y+7)$ <-- this equation establishes y as inverse of $\cos(3y+7)$

from here we solve for y

$x=\cos(3y+7)$ -> ... taking arccos of both sides
$\arccos(x)=\arccos\{\cos(3y+7)\}$ ->
$\arccos(x)=3y+7$
$\arccos(x)-7=3y$ ->
$[\arccos(x)-7]/3=y$

$$y=\frac{\arccos(x)-7}{3}$$ <-- This is an inverse of $\cos(3x+7)$

<u>Determining the Inverse of Composite Functions</u> (Method 2)

Determine an inverse of $y=\cos(3x+7)$. Make use of the fact that $f\{g(x)\}=x$, implies f is an inverse of g, to accomplish this.

Try to do this problem on your own before looking at the solution that follows. STOP

$\cos(3x+7)$ substituted into the inverse function f is x. Therefore if we build a function around $\cos(3x+7)$ that converts it to x. The function we build will be the inverse function of $\cos(3x+7)$

$$\frac{\arccos\{\cos(3x+7)\}-7}{3}=x$$

therefore

an inverse function of $\cos(3x+7)$ is $\dfrac{\arccos(x)-7}{3}$

<u>Determining Inverses of Composite Functions</u> (Method 3)

$f\{g(x)\}$ is a composite function composed of two functions, $g(x)$ inside of $f(x)$

<u>Theorem</u>

An inverse of the composite function $f\{g(x)\}$ is $g'\{f'(x)\}$ where $f'(x)$ is the inverse of $f(x)$ and $g'(x)$ is the inverse of $g(x)$.

Try to prove this theorem before looking at the proof below.

Proof

We make use of the fact that h{k(x)}=x implies h is an inverse of k.

g'{f'(f{g(x)}) } =
g'{g(x)} =
x

Proof Complete

Example problem

Make use of the fact that an inverse of f{g(x)} is g'{f'(x)}, (where g' the inverse of g and f' is the inverse of f) to calculate 'the' inverse of Cos(3x-7). Try to do this yourself before looking at the example below.

Cos(3x-7) is a composite function composed of 3x-7 inside of Cos. We begin by 'calculating' the inverse of 3x-7 and Cos. First we calculate an inverse of 3x-7 or y=3x-7

an inverse of 3x-7 = inverse of y=3x-7 which is x=3y-7 which is y=(x+7)/3.

Inverse of Cos is arccos.

Therefore an inverse of Cos(3x-7) is invcos(x) substituted into (x+7)/3 which is

$$y=\frac{\arccos(x)+7}{3} \quad \text{<---- inverse of Cos(3x-7)}$$

Inverse Trigonometry Functions Problem Set - part 2

1) Determine the inverse of each of the following functions. Make use of the fact that the inverse of y=f(x) is x=f(y) to do this. a) y=Cos(3x+7); b) y=Sin(1-x); c) y=Tan(2-5x)

2) Determine the inverse of each of the following functions. Make use of the fact that f{g(x)}=x implies that f is the inverse of g to do this. a) y=Cos(3x+7); b) y=Sin(1-x); c) y=Tan(2-5x)

3) [In this problem; assume; the inverse of f is f', the inverse of g is g'. Determine the inverse of each of the following functions. Make use of the fact that the inverse of f{g(x)} is g'{f'(x)} to do this. a) y=Cos(3x+7); b) y=Sin(1-x); c) y=Tan(2-5x).

4) Where p',q',r' are the inverses of p,q,r respectively, prove the inverse of the composite function p[q{r(x)}] is r'[q'{p'(x)}].

Note for problems 5 and 6 below: Some of these functions, unless they are suitably domain restricted do not have inverses. Let determining the required suitable domain restriction be an optional last step.

5.9.2) Determine an inverse of a suitably domain restricted (if required) form of each of the following functions.
a) cos{arcsin(x)};d b) arccos{Sin(x)}; c) sin{arccos(5x-2)};
d) arctan{Sin(ax-b)}

6) Determine an inverse of a suitably domain restricted (if required) form of each of the following functions.
a) arccos(wx-o); b) cos{1/(wt-a)}; c) 3*arctan(2x+8)-7°

Definition - Reflection of Point: Point A' is a reflection of point A, with respect to the a line, if segment AA' is perpendicular to the line and if the midpoint of AA' lies on the line.

Definition - Reflection of Function: (In this definition, reflections are with respect to a line). The functions f and f' are reflections of each other if ... 1) Every point of f' is a reflection of a point of f. 2) Every point of f is a reflection of a point of f'.

7) In this problem, the reflection is with respect to the line y=x.
a) Use classical geometry to prove the point (b,a) is the reflection of the point (a,b); b) Prove: The inverse function of any function (that has an inverse) is its reflection across the line y=x.

8) Determine the inverse of y=2x-3, then graph this function along with its inverse on the same graph. Does it appear as if the inverse of y=2x-3 is the mirror of y=2x-3 across the line y=x?

9) Extra Credit: a) Why does the following method work? To find the inverse of y=2x-5, substitute f(x) into 2x-5, and set this equal to x. Then solve for f(x), f(x) will equal the inverse of 2x-5. We do this here. 2f(x)-5=x -> 2f(x)=x+5 -> f(x)={x+5}/2, therefore the inverse of y=2x-5 is y={x+5}/2. b) Use this method to determine the inverse of the following functions
b) y=Cos(3x+7); c) y=1-2x; d) 4x+6y=5.

^^^^^^^^^^^^^^^^^^^^^^^^^^

10 Graphing Trigonometry Functions

^^^^^^^^^^^^^^^^^^^^^^^^^^

Graphing cos(t) and sin(t)

Make use of a compass to draw a circle on a piece of graph paper centered where two grids cross, (consider this the origin). Let this circle have a radius of 10 divisions, (each division represents a distance of 0.1). Put a small mark on the circle at 0, 15, 30, 45, 60, 75, and 90 degrees. Make use of this marked circle and the circle definition of cos(o) and sin(o) to estimate cos(o) and sin(o) of each of these angles to the nearest 1/100th. Record the values of cos(t) and sin(t) in a table. Make use of your knowledge of cos(o) and sin(o) to extend this table so that it includes the values of cos(o) and sin(o) for the angles, o= 0,15,30,45 ... 345 degrees. Use a calculator to verify the table you made is correct. Make use of this table, and graph paper to graph cos(o) and sin(o) over the domain 0° <= o < 360°. On the x axis let 1 division of the graph paper represent 15 degrees. On the y axis let each division represent of distance of 0.2. Make use of a graphing calculator or computer to verify your graphs are correct.

Note: Save the table and the graphs of cos(t) and sin(t). They will be used later in this chapter.

You have now graphed cos(x) and sin(x) over a domain of 360 degrees or one wavelength. cos(x) and sin(x) are periodic functions. Loosely speaking a periodic function is any function that repeats over its entire domain and the wavelength of a periodic function is the longest domain over which the function can be graphed before the graph begins a repeat. A function is a periodic function with a wavelength of L, if f(x)=f(x+L) (L>0), and L is the least positive number for which this is true. The wavelength of cos(o) and sin(o) is 2Pi radians or 360 degrees. Therefore if the domain of the graphs of cos(x) or sin(x) are extended beyond 360 degrees, their graphs begin to repeat.

-) Make use of a graphing calculator or computer to graph cos(x) from -2Pi to 4Pi so you can see the repetitive nature of cos(x).

When you graphed y=cos(o) and y=sin(o) over a domain of 1 wavelength, the domain of this graph going from 0 to 360 degrees corresponded to one revolution around the circle in a counter clockwise direction. Since one can travel around a circle an indefinite number of times in either direction, the domain of cos(t) and sin(t) is the entire x axis or from -infinity to +infinity.

If x is used as the domain variable of cosine or sine, the domain is considered to be distance. If t is used as the domain variable of cosine or sine, the domain is considered to be time. The domain of cos(x) is distance, the domain of cos(t) is time.

If the domain of a periodic function is time, not distance, it does not have a wavelength, but a period, which is loosely speaking the largest amount of time the function can go, before it begins a repeat. A function is a periodic function with a period of P, if f(t)=f(t+P) (P>0), and P is the least positive number for which this is true. As we have already seen, the definition for wavelength is similar.

If time is in seconds, the period of y=cos(t) and y=sin(t) is 2Pi seconds or approximately 6.283 seconds. Every 2Pi seconds the functions y=cos(t) and y=sin(t) start over and begin to repeat. The frequency of a time domain periodic function is the reciprocal of its period. Loosely speaking, frequency is how many times a periodic function repeats in a certain amount of time (typically a second). If (a,b) is a point of a periodic function f(x) such that no point of f(x) is farther away from the x axis than (a,b), then the distance of (a,b) to the x axis (or t axis) is the amplitude of f(x). The amplitude of y=cos(x) and y=sin(x) is 1. This is evident from the circle definition of each of these functions.

Note: When you graphed cos(x) and sin(x) by hand each division of the graph paper along the x axis represented 15 degrees, each division of the graph paper along the y axis represented a distance of 0.2. From here on, unless directed otherwise, the decision of how to scale graphs of trigonometry functions is up to you. You decide what each division of the graph paper represents. (Assuming you choose to use graph paper).

From here on, when plotting trigonometry functions, unless directed otherwise graph paper is optional. Don't plot numerous points as you did in the last graph, rather learn to plot trigonometry functions by drawing the waveform without the aid of numerous points. You will need to ensure the shapes of the functions are close to correct. Cosine and sine functions may be referred to as sinusoidals. When graphing a sinusoidal, all key points over a period of at least 1 period should be marked and labeled. Key points of a sinusoidal graph are function maximums, function minimums, the zeros of the function, i.e. where the function crosses the x axis, and where the function crosses the y axis. The preferred way to mark and label all key points is to draw a horizontal line that intersects all maximums, then label the y coordinate of this line; then do the same for all minimums. Then mark and label the x coordinates of; the zero's, the maximums and minimums; then mark and label the coordinate of the point where the sinusoidal crosses the y axis.

Graphing Problem Set 1

1) a) Study the graphs of y=cos(x) and y=sin(x) that you did previously, then put them away and do the following.
a) Graph y=cos(x) and y=sin(x) freehand from -2pi to 4pi. b) Mark and label all key points as instructed previously. c) Use a graphing calculator or computer to check your work.

2) What is the amplitude of a) cos(x)? b) 2*sin(x)? c) -6*cos(x)? d) A*sin(x)? e) -A*sin(x)? f) A*cos(t)? g) -A*cos(t)?

3) Graph each of the following on the same coordinate system from -180 degrees to 180 degrees). Use a different colored pencil for each graph. Mark and label all key points. Check your work using a graphing calculator or computer. a) y=cos(x); b) y=2cos(x); c) y=1/2 cos(x); d) y=-cos(x)

Compression and Expansion

Q: a) If the graph of y=cos(x) is stretched like an accordion along the x axis by a factor of 2, has it been compressed or stretched? b) If the graph y=cos(x) is stretched like an accordion along the x axis by a factor of 1/2, has it been compressed or stretched?

Compression Theorem

If any function y=f(x) is compressed horizontally towards the y axis by a factor of k, such that every point of the function becomes k times closer to the y axis, then the function becomes y=f(k*x).

For example, the period of the function y=cos(x) is 360 degrees or 2Pi radians. The function y=cos(k*x) has a period of 360/k degrees or 2Pi/k radians.

Proof

The function y=f(x) and the general point {a,f(a)} each represent (are) the set of points such that the y coordinate of any such point = f(the x coordinate of the point). Therefore y=f(x) and {a,f(a)} represent (are) the same set of points. The function y=f(k*x) and the general point {a/k,f(a)} each are the set of points such that the y coordinate of any such point = f(the x coordinate of the point * k). Therefore {a/k,f(a)} and y=f(k*x) are the same set of points. If the set of points {a,f(a)} is compressed horizontally towards the y axis such that each point becomes k times closer to the y axis, this set of points becomes {a/k,f(a)}. Therefore if the set of points y=f(x) is compressed horizontally towards the y axis such that each point becomes k times closer to the y axis, this set of points becomes y=f(k*x). Therefore any function y=f(x) compressed horizontally towards the y axis such that each of its points becomes k times closer to the y axis, becomes the function y=f(kx).

Proof Complete

Take the time to understand this theorem and its proof, then prove the compression theorem for yourself.

Note: From here on in this section, f(a*x) will be expressed as f(ax). The same goes for all the trigonometry functions.

Graphing Problem Set 2

1) Give a formal and informal definition for each of the following.
 a) Periodic Function b) wavelength; c) period; d) frequency

2) a) What is the wavelength of cos(x)? b) What is the period of sin(t)? c) What is the wavelength of sin(kx)? d) What is the period of cos(wt)? e) For what value of k does sin(kt) have a wave length of 20? f) For what value of w does cos(wt) have a frequency of 60?

3) Graph each of the following functions over a domain of two wavelengths (or periods) of your choosing. Give the wavelength, period, frequency and amplitude of each function. Mark and label all key points. Where applicable check your graphs using a graphing calculator or a computer. a) y=sin(2x); b) y=cos(t/3); c) y=sin(wt)

4) Give an equation for sinusoidals with each of the following sets characteristics and then graph these functions. Mark and label all key points.
 a) cosine function; amplitude = 1; wavelength = 2Pi
 b) sine function; amplitude = 2; wavelength = 1
 c) sine function; amplitude = 120; frequency = 60
 d) cosine function; amplitude = A; period = T
 e) sine function; amplitude = A; frequency = F

5) Prove: If a function y=f(x) is compressed towards the y axis by a factor of 'a', this function becomes y=f(ax).

6) Make use of what you learned by <u>proving</u> the theorem "If any function y=f(x) is compressed horizontally, about the y axis by a factor of k, such that every point of the function becomes k times closer to the y axis, then the function becomes y=f(kx)", to solve the following problems. a) If the function y=f(x) is shifted to the right by an amount of b, what does its equation become? b) If the function f(x+o) is compressed about the y axis such that each point becomes 'k' times closer to the y axis, what is the equation of this new function?

Shifting

Q: a) If the function f(x) is shifted to the right by 2, in what direction and by how much is it shifted? b) If the function f(x) is shifted to the right by -2, in what direction and by how much is it actually shifted?

Shifting Theorem

If any function f(x) is shifted to the right a distance of o, it becomes the function y=f(x-o).

For example if 'o' is 30 degrees, then the function cos(x-o) is the function cos(x) after it has been shifted 30 degrees to the right.

Proof

The function y=f(x) and the general point {a,f(a)} are each the set of points such that the y coordinate of any such point = f(the x coordinate of the point). Therefore y=f(x) and {a,f(a)} represent (are) the same set of points. The function y=f(x-o) and the general point {a+o,f(a)} each represent the set of points such that the y coordinate of the point = f(the x coordinate of the point - o). Therefore {a+o,f(a)} and y=f(x-o) are the same set of points. If the set of points {a,f(a)} is shifted to the right a distance of o, it becomes the set of points {a+o,f(a)}. Therefore if the set of points y=f(x) is shifted to the right a distance of o, it becomes y=f(x-o). Therefore any function y=f(x) shifted to the right a distance of o, becomes the function y=f(x-o).

Proof Complete

Take the time to understand this theorem and its proof, then prove the shifting theorem for yourself.

Graphing Problem Set 3

1) Graph each of the following functions over a domain of two wavelengths of your choosing. Mark and label all key points. Use a graphing calculator or a computer to verify your work.
a) y=cos(x-30°); b) y=2sin(t+60°); c) y=-2sin(x-Pi/2 rads);
e) y=cos(t-120°); f) y=sin(t+240°); g) y=cos(x+660°);
h) y=Acos(x-o)

2) Give an equation for the sinusoidals with the following characteristics and then graph these functions. Mark and label all key points. These sinusoidals have either a wavelength of 2Pi or a period of 2Pi.
   a) cosine function; amplitude = 1; shifted to left by Pi/3
   b) sine function; amplitude = 2; shifted to left by Pi/2
   c) sine function; amplitude = 5; shifted to right by Pi/3
   d) cosine function; amplitude = 3; shifted to right by 4Pi/3

3) Judging from the graphs of cos(x) and sin(x) it seems as if cos(x) shifted by some appropriate amount might be sin(x) and visa versa. a) Prove that cos(x) shifted by an appropriate amount is sin(x). b) Prove that sin(x) shifted by an appropriate amount is cos(x).

4) Prove: If a function y=f(x) is shifted to the right by 'b', the function becomes y=f(x-b).

Compression/Expansion and Shifting

We have learned that the k in y=cos(kx) changes the wavelength (or period if the domain is time). We have also learned that the 'o' in the function y=cos(x-o) shifts this function to the right or the left (depending if o is positive or negative). How do we graph a sinusoidal that has both of these changes? i.e. how do we graph the function y=cos(kx-o)? The following determination and discussion will answer this question.

Determination of how the -o in f(kx-o) affects this function

y=f(kx) and {a,f(ka)} are each the set of points such that the y coordinate of any such point = f(the x coordinate of the point * k). Therefore y=f(kx) and {a,f(ka)} are the same set of points. y=f(kx-o) and {a+o/k,f(ka)} are each the set of points such that the y coordinate of any such point = f(the x coordinate of the point * k - o). Therefore y=f(kx-o) and {a+o/k,f(ka)} are the same set of points. If the point {a,f(ka)} is shifted to the right a distance of o/k, it becomes the point {a+o/k,f(ka)}. Therefore if the function y=f(kx) is shifted to the right a distance of o/k, it becomes the function y=f(kx-o). Determination Complete

Take the time to understand this theorem and its proof, then prove this theorem for yourself.

We know that the -o in y=f(x-o) shifts this function 'o' to the right. We also know that the -o in y=f(kx-o) shifts this function o/k to the right. Therefore the function y=f(x) is shifted k times farther than the function f(kx) by the addition of the -o. However if these functions are periodic, adding -o to each of these functions shifts both of them by the same number of wavelengths. The k shortens the amount of shift by a factor of k, but it also shortens the wavelength by a factor of k.

Graphing Problem Set 4

1) Graph the following functions over a domain of 2 wavelengths of your choosing. Mark and label all key points. Then make use of a graphing calculator or computer to check your answers.
   a) y=3cos(2t-60°); b) y=2sin(3x+45°)
   c) y=-cos(4t+5Pi/6 rads); d) y=A*sin(bx+o) (A,b,o are positive).

2) Give an equation for each of the following sinusoidals. (Assume amplitudes are 1). Graph these functions. Make use of a graphing calculator or a computer to check your work. a) cosine function; stretched by a factor of 3 about y axis; then shifted to right by 30 degrees. b) sine function; compressed by a factor of 2 about y axis; then shifted left by 60 degrees. c) cosine function; shifted to the right by 30 degrees and then stretched by a factor of 3 about y axis. b') sine function; shifted to left by 60 degrees; then compressed by a factor of 4 about y axis.

3) Prove that the f(kx) shifted o/k to the right, is f(kx-o).

4) If the function y=f(kx) is shifted o/k to the right it becomes y=f(kx-o). a) Make use of 'this theorem' to derive the equation of the function f(x-o) after it has been compressed about the y axis by a factor of k. b) Derive the equation of the function y=f(x-o) after it has been compressed about the y axis by a factor of k. Do this using a method similar to how 'this theorem' was derived.

Graphing trigonometry functions with a negative domain variable

5.10.4) Graph the following functions over a domain of 2 wavelengths of your choosing. Mark and label all key points. Then make use of a graphing calculator or computer to check your answers.
   a) cos(Pi/6 rads-t); b) sin(Pi/6 rads-2t); c) 3cos(60°-2t);
   d) y=A*cos(o-bx) (A,b,o are positive).

Sinusoidals

A sinusoidal is a cosine function or a sine function, that is potentially: stretched or compressed horizontally, stretched or compressed vertically, shifted left or shifted right, or any combination of the aforementioned. Illustrating this: y=cos(x), y=sin(t), y=cos(t-a), y=sin(x-a), y=cos(wt), y=sin(wt), y=cos(wt-a) y=sin(wt-a) are all examples of sinusoidals. Any sinusoidal that is stretched or compressed horizontally, and/or stretched or compressed vertically and/or shifted to the left or to the right is (still) a sinusoidal.

Exercise) a) Give a definition of sinusoidal. b) Make use of this definition to give an equation of a general sinusoidal.

Relative Maximums and Relative Minimums of Functions

If a point 'p' of a function 'f' lies in the interior of a horizontal segment (meaning 'p' is a point of the segment but 'p' is not an end point of the segment), and if the y coordinate of this segment is greater than or equal to all y coordinates of the points of 'f' that have the same domain as this segment, then the y coordinate of point 'p' is a relative maximum of the function 'f'.

The definition of relative minimum is similar to the definition of relative maximum. It is left to the student make use of the definition relative maximum to construct a valid definition of relative minimum.

Hint: Before doing the problems below, graph y=cos(x)/|x| (-4Pi <= x <= 6Pi) using a graphing calculator or a computer.

1) In the explanation of what a relative maximum is, why is it necessary to state that 'p' lies in the interior of the segment?
2) How many maximums and minimums does y=cos(x) have?
3) How many relative maximums and relative minimums does y=cos(x) have?
4) How many maximums and minimums does y=cos(x)/|x| have?
5) How many relative maximums and relative minimums does y=cos(x)/|x| have?

Graphing Secant and Cosecant Functions

Next we will learn how to graph the secant and cosecant functions. Remember sec(o)=1/cos(o) and csc(o)=1/sin(o). In the discussion below a simple pattern how to graph these functions will be given. All possible situations will not be illustrated. To make the leap from the examples given and more general situations, will not be a big leap, but the student will be expected to use their reasoning to make this leap in problems that follow.

To graph y=sec(x), first graph y=cos(x) over the required domain. The function y=sec(x) will be graphed on the same coordinate system as y=cos(x). Where y=cos(x) is 1, y=sec(x) is also 1. Where y=cos(x) is positive and approaching 0, y=sec(x) is positive and is approaching +infinity. (This means as y=cos(x) gets closer and closer to 0, that y=sec(x) becomes higher and higher, there is no limit how high it gets). Where y=cos(x) is negative and approaches

0, y=sec(x) approaches negative infinity. Keep in mind that sec(x)=1/cos(x). Whenever you wish to graph any function approaching +infinity or -infinity, it is a good idea to draw an asymptote. An asymptote in this case this would be a vertical line at the places where cos(x) is 0. Once the asymptotes are drawn you can (more easily) graph y=sec(x), where it approaches positive infinity and where it approaches negative infinity.

When graphing a cosecant or secant function, all key points over a period of at least 1 period should be marked and labeled. Key points are, the points where the asymptotes cross the x axis, the relative maximums and relative minimums of the function, and the point where the function crosses the y axis (assuming x=0 is a part of the domain that is to be graphed). The preferred way to mark and label the key points is as follows. Draw a horizontal line that intersects all relative maximums, then label the y coordinate of this line; then do the same for all relative minimums. Then mark and label the x coordinates of; the points where the asymptotes cross the x axis. Mark and give the coordinate of the point where the secant (or cosecant) function crosses the y axis.

Graphing Problem Set 5

1) Graph the following functions over the domain from -2Pi radians to 2 Pi radians. Mark and label all key points. Make use of a graphing calculator to check your work. a) y=sec(x); b) y=csc(x); c) y=-sec(x); d) y=-csc(x)

2) Graph the following functions over a domain of 2 wavelengths of your choosing. Mark and label all key points of these functions. Then make use of a graphing calculator or computer to check your work. a) y=csc(x/2+60°); b) y=2sec[3(x-40°)];
c) y=sec(2x-2Pi/5 rad); d) y=csc(kx+o)

Graphing Tangent and Cotangent Functions

Graphing the tangent and cotangent functions is similar but somewhat more complex than graphing the secant and cosecant functions. To graph y=tan(x), first graph y=cos(x) and y=sin(x) (using different colors) on the same coordinate system over the required domain. tan(x) will be graphed on this same coordinate system (using a third color). Since tan(x)=sin(x)/cos(x), where cos(x)=sin(x), tan(x) equals 1. Also where sin(x) is 0, tan(x) is 0. Where y=cos(x) is approaching 0 and y=cos(x) and y=sin(x) are both positive or both negative, y=tan(x) approaches positive infinity. (Don't forget to draw the asymptotes). Where y=cos(x) is approaching 0 and the sign of sin(x) is the opposite of cos(x), y=tan(x) approaches negative infinity. Other values of tan(x) can be estimated keeping in mind that tan(x)=sin(x)/cos(x). Key points of the tangent and cotangent functions are; where the functional values are 1, the zeros of the function, where the asymptotes cross the x axis, and where the function intersects the y axis.

Graphing Problem Set 6

1) Graph the following functions over the domain from -2Pi radians to 2 Pi radians. Mark and label all key points. Make use of a graphing calculator to check your work. a) y=tan(x); b) y=cot(x); c) y=-tan(x); d) y=-cot(x)

2) What is the wavelength of a) y=tan(x); b) y=cot(x); c) y=tan(kx)

3) Graph the following functions over a domain of 2 wavelengths of your choosing. Mark and label all key points of these functions. Then make use of a graphing calculator or computer to check your work. a) y=tan(2x+pi/3 rad); b) y=cot[3(x-30°)]; c) y=-cot(b*x+o) (b and o are positive)

4) Graph the following over a domain of at least 2 wavelengths. a) tan(90°-2x); b) cot(a-kx) (assume a and k are positive)

Graphing Inverse Trigonometry Functions

If y=f(x) is a function that has an inverse. The inverse of y=f(x) can be obtained by exchanging the x and the y, i.e. x=f(y) is the inverse of y=f(x). The student proved this in the Inverse Trigonometry functions section and the proof is also provided in the answers section. Also if (a,b) is a point of a function y=f(x), then (b,a) is a point of the inverse function x=f(y). This too was previously proven.

We learned in the Inverse Trigonometry Functions section that the domain restricted trig functions,

y=Cos(o), y=Sin(o), y=Tan(o), y=Sec(o), y=Csc(o), y=Cot(o)

and the functions

y=arccos(x), y=arcsin(x), y=arctan(x), y=arcsec(x), y=arccsc(x) y=arccot(x)

are inverses of each other respectively. Therefore, if (a,b) is a point of y=Cos(x), then (b,a) is a point of y=arccos(x) and visa versa. This applies to all (domain restricted trig) functions and their inverses. We can make use of this fact to graph inverse (trig) functions.

To graph an inverse (trigonometry) function, first graph the function f(x). Then identify a set of points of the function necessary to recreate a good picture of this function, assuming you are going to recreate the function by connecting these points with a smooth curve. this set of points should include the key points of the function. Next on another piece of graph paper, plot the set of points (flipped about the y axis). Connect these points with a smooth curve. This graph is the graph of the inverse (trigonometry) function.

When graphing an inverse trigonometry function, key points should be marked and labeled. To identify key points of an inverse trigonometry function, first identify key points of the corresponding trigonometry function. These points, after their x and y coordinates have been transposed are the key points of the inverse trigonometry function. For example, if (a,b) is a key point of a trigonometry function, then (b,a) is a key point of the corresponding inverse trigonometry function.

Note: Remember the domains of Cos(o), Sin(o) and Tan(o) are restricted as follows: Cos(o) (0°<= o <=180°);
Sin(o) (-90°<= o <=90°); Tan(o) (-90° <= o <= 90°). The domains of Sec(o), Csc(o) and Cot(o) are the same as domain restricted trig functions that they correspond to.

Graphing Problem Set 7

1) Graph y=arccos(o) using the following method. Make a list of points of y=Cos(o) for o=(0°,15°,30° ... 180°). Make use of this list of points to make a list of points of the function y=arccos(o). This second list of points is the same as the first list of points where the x and y coordinates are exchanged with each other. Then make use of these points to graph y=arccos(o). Make use of a graphing calculator or computer to check your work.

2) Using the following method, graph each of the following inverse trig functions. Then make use of a graphing calculator or computer to check your work. ... Graph the corresponding domain restricted trig function, mark and label all key points. If applicable draw the asymptotes. Now (if applicable) draw the asymptotes of the inverse trig function, {If the point (a,b) is a point of an asymptote of a trig function, then (b,a) is a point of an asymptote of the corresponding inverse trig function}. Make use of the key points of the trig function to make a list of the key points of the corresponding inverse trig function. Make use of these key points and the asymptotes of the inverse trig function (if they exist) to graph the inverse trig function.
a) y=arccos(x); b) y=arcsin(x); c) y=arctan(x); d) y=arcsec(x);
e) y=arccsc(x); f) y=arccot(x); g) y=-arccos(x); h) y=-arccot(x)

3) In the inverse trig functions section, you learned how to simplify the following functions. You can make use of this in graphing these functions (or not). Graph the inverse of
a) Cos(2x-60°); b) Sin(x-30°)

4) Graph the inverse of the following functions, mark and label all key points. a) sec(x+30°); b) csc(x-30°); c) cot(x+45°)

5) Analytically determine the inverse of y=Cos(2x+1). Make use this result to graph the inverse of y=Cos(2x+1) on graph paper. Graph y=Cos(2x+1) on the same graph. Graph the line y=x on the same graph. If you did this correctly you should notice that y=Cos(2x) and its inverse are reflections of each other across the line y=x. If not, evaluate what went wrong and do correctly.

^^^^^^^^^^^^^^^^^^^^^^^^^^

11 Sinusoidal Multiplication and Addition

^^^^^^^^^^^^^^^^^^^^^^^^^^

The topics of this section are perhaps the most interesting and useful in Trigonometry. This is saying a lot, there is stiff competition for interesting and useful from other topics of Trigonometry.

If any two sinusoidals of different frequency are multiplied by each other the result will be a function which can also be expressed as two other sinusoidals which are added together. [This is a rather amazing geometric fact]. If a sinusoidal with a frequency of A is multiplied by a sinusoidal of frequency B, the result will be a function that can also be expressed as the sum of two other sinusoidals, one with a frequency of A+B and the other with a frequency A-B. Electrical engineers make use of this property in the design of AM radios and transmitters.

Another interesting property of sinusoidals is that if any two sinusoidals of the same wavelength (or frequency) are added together the result is a sinusoidal of this same wavelength (or frequency). For example 4sin(3x)+cos(3x+17°) is a sinusoidal, of the same frequency as 4sin(3x) and cos(3x+17°).

Another interesting property of sinusoidals is if any two sinusoidals of the same wavelength or frequency are multiplied together, the resultant function is a vertically displaced sinusoidal. For example cos(2t)*cos(2t-17°) is a vertically displaced sinusoidal.

All these interesting properties and more will be proven and discussed in this section.

Preliminary Exercises

At this point it is important for students to have an intuitive grasp of what it means to add two functions together, and what it means to multiply two functions together. To ensure this is so, the following two preliminary problems are given.

I) Graph y=cos(2t)+sin(3t), do this by graphing y=cos(2t) and y=sin(3t) from 0 to 2Pi on the same graph space using different colored pencils. Graphically add these two functions together at x=0, Pi/6, 2Pi/6 ... 12Pi/6 = 2Pi. Estimate point locations. Connect these points with a smooth curve using a third colored pencil. Check your graph using a graphing calculator or computer.

II) Do not plot points in this exercise: Graph y=cos(8t)cos(t). Do this by graphing y=cos(t), then y=cos(8t) from 180° to 180° in the same graph space using different colored pencils. Graphically multiply these two functions together by estimating and then plotting the graph of y=cos(8t)cos(t). Use a third colored pencil. Check your graph using a graphing calculator or computer. (Don't spend a lot of time making your graph highly accurate, approximation is good enough, its the understanding that counts).

Modulation: A sinusoidal is said to be modulated when it is multiplied by another function. The basic sinusoidal multiplication (modulation) identities and their proofs follow.

Basic Sinusoidal Multiplication Identities

$$\cos(a)\cos(b)=1/2[\cos(a+b)+\cos(a-b)]$$
$$\sin(a)\sin(b)=1/2[\cos(a-b)-\cos(a+b)]$$
$$\sin(a)\cos(b)=1/2[\sin(a+b)+\sin(a-b)]$$

cosine-cosine modulation identity

$$\cos(a)\cos(b)=\frac{1}{2}\{\cos(a+b)+\cos(a-b)\}$$

Proof

$$\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$$
$$\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$$

adding these two equations together we get

$$\cos(a+b)+\cos(a-b)=2\cos(a)\cos(b)$$

rearranging this equation we get

$$\cos(a)\cos(b)=\frac{1}{2}\{\cos(a+b)+\cos(a-b)\}$$

Proof complete

Exercise

a) Make use of the cosine cosine modulation identity to give an alternate form of the function cos(2t)cos(3t). b) Make use of a graphing calculator or computer to graph y=cos(2t)cos(3t) and the function you calculated in part 'a'. c) Compare both graphs, are they the same?

In this section we are presenting and then proving the three basic modulation identities. However only one modulation identity is necessary in practice. This is because any sinusoidal can be represented as a cosine function (or a sine function) function that is appropriately shifted to the left or right. The reason all three basic modulation identities are presented here is for completeness sake and also so that students have more opportunity to see and to practice proving modulation identities. In electrical engineering, a variant of the cos(a)cos(b) modulation identity that we just proven is the only one that is used, this is because it is the simplest.

sine-sine modulation identity

$$\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$$

Proof

$$\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$$
$$\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$$

subtracting the second equation from the first we get

$$\cos(a-b)-\cos(a+b)=2\sin(a)\sin(b)$$

rearranging this equation we get

$$\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$$

proof complete

cosine-sine modulation identity

$$\sin(a)\cos(b)= \frac{1}{2}[\sin(a+b)+\sin(a-b)]$$

Proof

sin(a+b)=sin(a)cos(b)+sin(b)cos(a)
sin(a-b)=sin(a)cos(b)-sin(b)cos(a)

adding these two equations together we get

sin(a+b)+sin(a-b)=2sin(a)cos(b)

rearranging this equation we get

sin(a)cos(b)= 1/2[sin(a+b)+sin(a-b)]

proof complete

Example

Lets make use of the cosine-cosine modulation identity cos(a)cos(b)=1/2{cos(a+b)+cos(a-b)} to determine what happens if the function cos(7t) is modulated by the function cos(2t). In this equation we substitute 7t into 'a' and 2t into 'b'. This gives the equation cos(7t)cos(2t) = 1/2{cos(7t+2t)+cos(7t-2t) = 1/2{cos(9t)+cos(5t)} = 1/2*cos(9t) + 1/2*cos(5t).

so cos(7t)cos(2t) = 1/2*cos(9t) + 1/2*cos(5t)

Beats

Make use of a graphing calculator or computer to graph the function y=cos(12t)cos(t) from t=0 to t=2Pi. Refer to this graph in the discussion below.

The graph of y=cos(12t)*cos(t) looks like the graph of y=cos(12t) except that the amplitude of y=cos(12t)*cos(t) varies, and the amplitude of y=cos(12t) is constant. [keep in mind the amplitude of cos(12t) is 1]. We define the amplitude of y=cos(12t)cos(t) as |cos(t)|. Keep in mind that cos(t) has a much longer wave length than cos(12t), (12 times as long). Since cos(t) crosses the x axis (becomes 0) every Pi radians or 180 degrees, every 180 degrees the amplitude of cos(12t)*cos(t) is 0.

The function of y=cos(12t)*cos(t) between any two consecutive zeros of cos(t) is one beat of the function y=cos(12t)*cos(t). From t=0 to t=Pi is one beat of y=cos(12t)*cos(t). From t=Pi to t=2Pi is another beat of y=cos(12t)*cos(t). Please refer to the graph of y=cos(12t)cos(t) that you did so that the concept of what a beat is becomes clear in your mind.

It should be noted that consecutive beats of y=cos(12t)*cos(t) are not identical, they are 'opposite functions' of each other. Why is this? However any beat of y=cos(12t)*cos(t) is identical to a beat that is two beats away from it. Inspite of this, all the beats look pretty much alike.

The function that was created by multiplying cos(12t) and cos(t) could also have been created by adding the functions 1/2*cos(16t) and 1/2*cos(14t). This according to the identity cos(a)cos(b)=1/2[cos(a+b)+cos(a-b)]. One method of tuning musical instruments relies on the fact that beats are created when sounds which are close in frequency, but not exactly the same frequency are added together. A tuning fork is a device which emits sinusoidal sound waves at a particular frequency. Suppose a note of a particular instrument is supposed to emit a sound at 440 Hz. (This is middle C). If we wish to determine if the note is tuned correctly (emitting the correct frequency), we play the note and a tuning fork that emits sinusoidal sound waves at 440 Hz. In the air these two sounds are added together. If the instrument is close to 440 Hz but not exactly 440 Hz, the instrument note together with the sound of the tuning fork together will create beats which can be heard. These beats sound like a note very close to middle C where the sound will get louder then softer then louder then softer etc. These are the beats. If beats are heard, the instrument is adjusted and the process is repeated until no beats are heard. At this point the instrument is properly tuned.

A "beated sinusoid" can be thought of as two sinusoids added together, or as two sinusoids multiplied together. Each description though seemingly very different refers to the same reality. This is not unprecedented in mathematics. The induction axiom can be stated, "for every integer these is a next higher integer". It can also be stated, "every set of positive integers, contains a least integer". This last form of the induction axiom is referred to as the 'Well Ordered Principal'. These seemingly very different statements describe the same reality. It can be proven that these two different forms of the induction axiom are mathematically equivalent.

Preliminary Problems

1-) A sinusoidal is a cosine function, a sine function, (or any other sinusoidal) that is potentially; stretched or compressed horizontally, stretched or compressed vertically, shifted left or shifted right. Make use of this definition to write the equation of a general sinusoidal in a) cosine form; b) sine form.

2-) Assuming time is in seconds, what is the frequency of a) cos(t)? b) cos(wt)? c) If cos(wt) has a frequency of 843Hz, what is w?

Sinusoidal Multiplication Problem Set

1) Prove cos(a)cos(b)=1/2[cos(a+b)+cos(a-b)]
2) Prove sin(a)sin(b)=1/2[cos(a-b)-cos(a+b)]
3) Prove sin(a)cos(b)=1/2[sin(a+b)+sin(a-b)]

4) The following functions are expressed as two sinusoidals which are multiplied together. Express them as two sinusoidals which are added together. a) y=sin(4t)*sin(7t); b) y=cos(t)*cos(2t)

5.11.1) Make use of cos(a)cos(b)=1/2[cos(a+b)+cos(a-b)] to derive sin(a)sin(b)=1/2[cos(a-b)-cos(a+b)].

6) Make use of cos(a)cos(b)=1/2[cos(a+b)+cos(a-b)] to derive sin(a)cos(b)=1/2[sin(a+b)+sin(a-b)].

7) Using a graphing calculator or computer, graph y=cos(t)cos(8t) from -180 to 360 degrees. Can you see the beats?

8.11.1) The following functions are expressed as two sinusoidals which are added together. Express them as two sinusoidals which are multiplied together. a) y=cos(3t)+cos(8t); b) y=cos(2t)-cos(3t); c) y=cos(at)+cos(bt); d) y=sin(at)+sin(bt)

9.11.1) Prove that any sinusoidal multiplied by any other sinusoidal of a different wavelength or frequency can be expressed as two other sinusoidals which are added together.

10) Prove that the following sinusoidal products form a single vertically displaced sinusoidal and not two sinusoidals which are added together. a) cos(t)cos(t); b) cos(wt)sin(wt); c) sin(wt)sin(wt). d) Can you explain what happened to the other 'missing sinusoidal' in each of these problems? Hint: What is the wave length of y=cos(z*t)?

11.11.1) Prove: Any sinusoidal multiplied by any other sinusoidal of the same wavelength or frequency is a vertically displaced sinusoidal.

12.11.1) a) Make use of the cosine-cosine modulation identity

$$\cos(a)\cos(b)= 1/2 \cos(a+b) + 1/2 \cos(a-b)$$

to derive

$$\cos(ax-m)\cos(bx-n)= 1/2 \cos\{(a+b)x-(m+n)\} + 1/2 \cos\{(a-b)x-(m-n)\}$$

b) Make use of the equation derived in 'a', to prove

(Any sinusoidal of frequency U)*(Any sinusoidal of frequency V)=

{A sinusoidal of frequency (U+V)}+
{A sinusoidal of frequency (U-V)}

Radio Theory, A Trigonometry Perspective

Almost any function can be decomposed into sinusoidals of various wavelengths or frequencies which are added to one another. [This is another amazing geometrical fact having to do with sinusoidals]. How to do this is beyond the scope of this book, for more information see Fourier transform and Fourier integral. Many common functions consist of an infinite number of sinusoidals added together. Such functions can be closely approximated by adding together a finite subset of this infinite set of these sinusoidals. The more sinusoidals used, the closer the approximation will be.

All sounds, including a human voice speaking words are examples of functions that can be decomposed into sinusoidals which are added together. Human voices can be decomposed into a summation of sinusoidals ranging in frequency from about 85Hz to about 3400Hz].

A sinusoidal is said to be modulated when it is multiplied by another function. When a person speaks into a microphone at an AM radio station, a microphone converts the voice of the person speaking into an electrical signal. Electronic circuitry then multiplies this signal with a sinusoidal whose frequency is about 1 MHz (1 million cycles per second). This approximately 1MHz sinusoidal is referred to the carrier wave of the radio station. It is the carrier wave modulated by the signal of the persons voice that gets sent to the radio station antenna so it can be transmitted 'over the air'.

The modulation equations we derived in this section deal with modulation of a sinusoidal by another sinusoidal. They do not deal with modulation of a sinusoidal by a general function. However the fact that most functions are composed of sinusoidals added together means we can deal with modulation of a sinusoidal by a general function also.

Example For simplicity sake, suppose the signal (the function) of a person's voice is composed of the following sinusoidals added together. y=cos(3,000t), y=cos(6,000t) and y=cos(12,000t). Suppose also this voice signal modulates the carrier wave cos(100,000t). This modulated carrier can be expressed as sinusoidals added together. What are these sinusoidals?

We have cos(100,000t) being modulated by [cos(3,000t)+cos(6,000t)+cos(12,000t)] which is

cos(100,000t)[cos(3,000t) + cos(6,000t)+cos(12,000t)] =

cos(100,000t)*cos(3,000t) + cos(100,000t)*cos(6,000t) +
cos(100,000t)*cos(12,000t) =

[Making use of the identity cos(a)cos(b)=1/2[cos(a+b)+cos(a-b)]

1/2*cos(97,000t) + 1/2*cos(103,000t) +
1/2*cos(94,000t) + 1/2*cos(106,000t) +
1/2*cos(88,000t) + 1/2*cos(112,000t)

If a signal can be decomposed into sinusoidals, the bandwidth of the signal is the frequency of the sinusoidal with the highest frequency minus the frequency of the signal with the lowest frequency.(use Hz)

If a signal can be decomposed into sinusoidals, the spectrum of the signal is the set of frequencies of all the sinusoidals that make up the signal. {Use hertz (Hz), which means cycles per second}.

[A one dimensional graph, consists of a line where the parts of the line that are of interest are darkened and and the numerical values of the endpoints of the darkened portions are given].

Exercise: Referring to the previous example, a) What is the bandwidth of a person's voice? b) What is the bandwidth of the signal that gets sent to the antenna to be transmitted 'over the air'? c) What is the spectrum of a persons voice? d) What is the spectrum of the signal that gets sent to the antenna to be transmitted 'over the air'? e) On a one dimensional graph, graph the spectrum of the persons voice. f) On this same 1 dimensional graph, graph the spectrum of signal that gets transmitted (the modulated carrier). (This graph needn't be to scale).

In the previous exercise notice that half of the sinusoidals that comprise the modulated carrier, (which gets sent to the antenna to be transmitted), have a frequency less than the frequency of the carrier. The other half of these sinusoidals have a frequency greater than the frequency of the carrier. That half of the sinusoidals which have a frequency less than the carrier frequency is referred to as the lower side band (LSB). That half of the sinusoidals which have a frequency greater than the carrier

frequency is referred to as the upper side band (USB).
In a regular AM radio broadcast, the entire modulated carrier wave or both side bands are transmitted by the radio station and then received by the radio. Ham radio [learn more about ham radio on the internet or at a library] makes use of regular AM transmission and reception, but also makes use of (AM) single side band (SSB) transmission and reception. In single side band transmission, electronic circuits filter out either the lower or the upper side band and then transmit the side band that does not get filtered out. The radio receiver makes use of the side band that is received, along with knowledge of the carrier frequency to reconstruct the missing side band and add it back in, this recreates the modulated carrier. The radio is given 'knowledge' of the carrier frequency as the radio operator tunes the radio .. by turning the tuning dial.

One advantage of single side band or SSB is that it uses less spectrum. Given that there is only a certain amount of spectrum available, this is a pretty good advantage. Another advantage of SSB compared to AM is that a signal transmitted with a given amount of power can be picked up further away than if the signal were a regular AM signal. The disadvantage of SSB radio transmission is that transmitters and radios that make use of it cost more to make.

Radio Theory Problem Set

0) Ensure you do the exercise on the previous page.

1) Explain how it is that a carrier wave of a radio station, when modulated by a persons voice creates a signal with two side bands, a lower side band which consists of sinusoidals added together whose frequencies are lower than the carrier wave, and an upper side band consisting of sinusoidals added together whose frequencies are higher than the carrier wave.

2) In reality a human voice consists of an infinite set of sinusoidals ranging in frequency from about 85Hz to about 3400Hz. [To understand this, read about the Fourier integral]. The spectrum of a human voice is continuous, every frequency greater than the lowest frequency and less than the highest frequency is present within the voice. a) Using a one dimensional graph, graph the spectrum of a human voice. b) How much bandwidth does a human voice have? Assume a human voice is to be transmitted on an AM radio station whose carrier wave has a frequency of 1MHz. c) Graph and label the spectrum of this transmitted signal. (The graph needn't be to scale). d) What is the bandwidth of this transmitted signal? e) Assuming LSB (Lower Sideband) Transmission is used instead of AM transmission, repeat problems 'c' and 'd' above.

One of the useful things modulation accomplishes is to shift the spectrum of the persons voice that is to be transmitted over 'the air' to a new location. For example the spectrum of a human voice is located from about 85Hz to about 3400Hz. It would be possible for a radio station to broadcast the actual signal of a persons voice without modulating it. [Although for efficient broadcast and reception, doing this would require use of a very long antenna]. If radio stations broadcast the actual signal, such as a persons voice without modulating it, there could be only one radio station in a given area, because if there were more than one, a receiver would pick up all the radio stations in that area because they are all broadcasting over the same spectrum or set of frequencies. This would sound like many people in a room talking at the same time. To avoid this situation, each radio station in a given area moves the voice they transmit to a different spectrum location by modulating it with a carrier of a different frequency. 910kHz, 960kHz, 1010kHz, and 1010kHz are examples of frequencies assigned for use by commercial AM radio stations in Salt Lake City, Utah area. Another city far such as Denver Colorado may use these same carrier frequencies, because with enough distance between radio stations, they will not interfere with each other.

A radio tunes into to any particular station by electronically filtering out all frequencies or spectrum except that which it wants to listen to. Electronic circuits in the radio filter out all unwanted frequencies. Electronic circuitry in the radio then demodulate the modulated signal which was broadcast by the radio station leaving only the electrical signal of the human voice before it was modulated at the radio station. This signal is then sent to the speaker of the radio where it is converted to sound.

Q: Give two reasons why signals are modulated before they are transmitted 'over the air' by a radio station.

To learn more about filter and radio theory and design, study Electrical Engineering at a University. You can help lay the foundation for doing this by taking an AP Physics class in High School.

Basic Sinusoidal Addition Identities

The basic sinusoidal addition identities are

$\cos(a)+\cos(b)=2\cos\{(a+b)/2\}\cos\{(a-b)/2\}$
$\sin(a)+\sin(b)=2\sin\{(a+b)/2\}\cos\{(a-b)/2\}$
$\cos(a)+\sin(b)=[\cos\{(a-b)/2\}-\sin\{(a-b)/2\}][\cos\{(a+b)/2\}+\sin\{(a+b)/2\}]$

Note: The right side of cos(a)+sin(b)= .. can also be expressed as 2 sinusoidal multiplied by each other, i.e. $[\cos\{(a-b)/2\}-\sin\{(a-b)/2\}]$ and $[\cos\{(a+b)/2\}+\sin\{(a+b)/2\}]$ can each be expressed as sinusoidals. In a future problem students will be asked to prove this, a solution will also be provided.

Their proofs follow

Theorem

$$\cos(a)+\cos(b)= 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Proof

$\cos(a)+\cos(b) =$

------

we make the following substitutions

$a=u+v$ and $b=u-v$; implying that $u= \frac{1}{2}(a+b)$ and $v= \frac{1}{2}(a-b)$ (verify this)

$\cos(u+v)+\cos(u-v) =$
$\cos(u)\cos(v)-\sin(u)\sin(v) + \cos(u)\cos(v)+\sin(u)\sin(v) =$

$2\cos(u)\cos(v) =$

$$2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Proof Complete

Theorem

$$\sin(a)+\sin(b)= 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Proof

$\sin(a)+\sin(b)=$

------

we make the following substitutions

$a=u+v$ and $b=u-v$; implying that $u= \frac{1}{2}(a+b)$ and $v= \frac{1}{2}(a-b)$

$\sin(u+v)+\sin(u-v)=$
$\sin(u)\cos(v)+\sin(v)\cos(u) + \sin(u)\cos(v)-\sin(v)\cos(u) =$

$2\sin(u)\cos(v) =$

$$2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Proof Complete

The following theorem like the two previous ones, can be used to show that two particular sinusoidals added together that can be expressed as two other sinusoidals which are multiplied together. Each of the two terms on the right side of the following identity are a single sinusoidal, even though each is expressed as the sum of two sinusoidals. The reason each of these two terms are a single sinusoidal, is ...

If any two sinusoidals having the same amplitude and, wavelength or frequency, are added together the resultant function is a single sinusoidal.

A problem will be given in the next section asking students to prove this. The solution to this problem will also be given.

Theorem

$$\cos(a)+\sin(b)=\left\{\cos\left(\frac{a-b}{2}\right)-\sin\left(\frac{a-b}{2}\right)\right\}\left\{\cos\left(\frac{a+b}{2}\right)+\sin\left(\frac{a+b}{2}\right)\right\}$$

Proof

$\cos(a)+\sin(b)=$

------

we make the following substitutions

$a=u+v$ and $b=u-v$; implying that $u=\frac{1}{2}(a+b)$ and $v=\frac{1}{2}(a-b)$

$\cos(u+v)+\sin(u-v)=$

$\cos(u)\cos(v)-\sin(u)\sin(v) + \sin(u)\cos(v)-\sin(v)\cos(u)=$
$\{\cos(v)-\sin(v)\}\{\cos(u)+\sin(u)\}=$

$$\left\{\cos\left(\frac{a-b}{2}\right)-\sin\left(\frac{a-b}{2}\right)\right\}\left\{\cos\left(\frac{a+b}{2}\right)+\sin\left(\frac{a+b}{2}\right)\right\}$$

Proof Complete

In the next problem set (problem 11), a problem (and its solution) is given asking the student to prove that

$$\left\{\cos\left(\frac{a-b}{2}\right)-\sin\left(\frac{a-b}{2}\right)\right\}\left\{\cos\left(\frac{a+b}{2}\right)+\sin\left(\frac{a+b}{2}\right)\right\} =$$

$$2\cos\left(\frac{a+b}{2} - 45^\circ\right)\cos\left(\frac{a-b}{2} + 45^\circ\right)$$

which completes the process of showing that the two sinusoidals cos(a) and sin(b) added together are also two other sinusoidals multiplied together, just as is the case with cos(a) + cos(b) and sin(a) + sin(b).

Sinusoidal Addition Problem Set

Do Problems 1,2,3 using the same proofs given previously.

1) Prove cos(a)+cos(b)= 2cos{(a+b)/2}cos{(a-b)/2}
2) Prove sin(a)+sin(b)= 2sin{(a+b)/2}cos{(a-b)/2}
3) Prove cos(a)+sin(b)=
[cos{(a+b)/2}+sin{(a+b)/2}] * [cos{(a-b)/2}-sin{(a-b)/2}]

4.11.2) Express each of the following as a product of sinusoidals.
a) cos(10t)+cos(8t); b) sin(5t)+sin(3t); c) cos(2t)+sin(3t);
d) 1+sin(t).

5.11.2) Make use of cos(a)+cos(b)=2cos{(a+b)/2}cos{(a-b)/2} to derive sin(a)+sin(b)=2sin{(a+b)/2}cos{(a-b)/2}
the equation you derived in problem #2 using another method.

6) a) Make use of
cos(a)+cos(b)=2cos{(a+b)/2}cos{(a-b)/2} to derive
cos(a)+sin(b)= 2cos{(a+b)/2 - 45°}cos{(a-b)/2 + 45°}
b) Make use of the equation derived in part 'a' to derive
cos(a)+sin(b)=
[cos{(a+b)/2}+sin{(a+b)/2}] * [cos{(a-b)/2}-sin{(a-b)/2}],
the equation you derived in problem #3 using another method.

7) Using a method of your choice derive
cos(a)-cos(b)= -2sin{(a+b)/2}sin{(a-b)/2}

8.11.2) Derive sin(a)-sin(b) = 2cos{(a+b)/2}sin{(a-b)/2}

9) a) Derive cos(a)-sin(b) = [cos{(a+b)/2}-sin{(a+b)/2}] *
[cos{(a-b)/2}+sin{(a-b)/2}]
b*) Derive the equation for sin(a)-cos(b)= ...

10.11.2) Prove: Any sinusoidal added to any other sinusoidal of the same amplitude, and wavelength (or frequency) is a single sinusoidal of that same wavelength or frequency.

11.11.2) Make use of cos(a)+sin(b)=
[cos{(a+b)/2}+sin{(a+b)/2}][cos{(a-b)/2}-sin{(a-b)/2}]
to derive cos(a)+sin(b)= 2cos{(a+b)/2 - 45°}cos{(a-b)/2 + 45°}

12) a) Prove: If any sinusoidal with a frequency of U is added to any other sinusoidal of the same amplitude, and a frequency of V, the result is a sinusoidal of frequency (U+V)/2 multiplied by a sinusoidal of frequency of (U-V)/2.

b) Make use of the work you did in part 'a' to solve the following problem. Two tuning forks of the same loudness (amplitude) are struck. One emits a sinusoidal note of 318Hz, the other emits a sinusoidal note of 322Hz. a) What is the frequency of the note that these tuning forks together emit? b) What is the period of the beats?

Note: The two tuning forks together emit is a single sinusoidal that is constantly changing phases (by 180°)(and is modulated). However psychological research has shown that humans are not able to distinguish between sounds where the only difference between the sounds is are phase differences.

<u>A*cos(wt)+B*sin(wt)=C*cos(wt-o)</u>

Previously we proved that the sum of <u>any</u> two sinusoidals having the same amplitude, and wavelength or frequency is a sinusoidal of that same wavelength or frequency. We will now show that the sum of <u>any</u> non shifted cosine function, and <u>any</u> non shifted sine function of the same wavelength or frequency, regardless of amplitude, is a sinusoidal of that same wavelength or frequency.
To do this we must make use of any of the following identities.

cos(a-b)=cos(a)cos(b)+sin(a)sin(b)
sin(a+b)=sin(a)cos(b)+cos(a)sin(b)

cos(a+b)=cos(a)cos(b)-sin(a)sin(b)
sin(a-b)=sin(a)cos(b)-sin(a)sin(b)

We will demonstrate by showing that 3cos(t)+4sin(t) is a sinusoidal. To do this we choose to make use of

sin(a+b)=sin(a)cos(b)+sin(b)cos(a) ... (Why?)

we multiply both sides of this equation by A (A>0) and substitute t into b which is,

A*sin(a+t)=A*sin(a)cos(t)+A*sin(t)cos(a)

rearranging and setting this equation equal to 3cos(t)+4sin(t) gives

eq 1:
3cos(t)+4sin(t)= [A*sin(a)]cos(t)+[A*cos(a)]sin(t)= A*sin(a+t)

Therefore A*sin(a)=3 and A*cos(a)= 4 --->

--We now solve for sin(a)--

```
A*sin(a)   3    sin(a)          sin(a)
------- = --- = ------ = ------------------- --->
A*cos(a)   4    cos(a)   +/- sqrt{1-sin^2(a)}
```

4sin(a) = +/- 3sqrt(1-sin^2(a)} --->

16sin^2(a) = 9{1-sin^2(a)} --->

16sin^2(a) = 9-9sin^2(a)} --->

25sin^2(a) = 9 --->

5sin(a) = +/- 3 --->

```
              3
sin(a)= +/-  ---
              5
```

We now solve for A

Asin(a)=3 --->   [this equation was given previously]

```
      3         3
A = ------ = ------- =  +/- 5, but (A>0) so  ...  eq 3: A= 5
    sin(a)        3
              +/-  -
                   5
```

solving now for cos(a)

```
                                         {    9 }           {16}
cos(a) =  +/- sqrt{1-sin^2(a)} = +/- sqrt{1- ---} = +/- sqrt{--} =
                                         {    25}           {25}
      4
 +/-  -   so
      5
             4
cos(a)= +/-  -
             5
```

so A=5; sin(a)= +/- 3/5 ; cos(a)= +/- 4/5; from

eq 1:
3cos(t)+4sin(t)= [A*sin(a)]cos(t)+[A*cos(a)]sin(t)= A*sin(a+t)

we have A*sin(a)=3; A*cos(a)=4 which implies

$\cos(a)=\frac{4}{5}$ and $\sin(a)=\frac{3}{5}$ --->

an approximate value for 'a' is 36.87.. degrees.

Therefore (referring to equation 1)

3cos(t)+4sin(t)=5sin(t+36.87..°)

Sinusoidal Addition Problem Set (part 2)

1) Decompose each of the following into a non shifted cosine function added to a non shifted sine function. a) y=cos(wt-30°); b) y=cos(wt+45°); c) y=cos(wt-o); d) y=sin(wt-o)

2) Express each of the following as a shifted cosine function. a) y=2cos(t)+3sin(t); b) 7sin(t)-5cos(t).

When doing problem the next problem, do not make use of the work you did when doing the previous problem.

3) Express each of the following as a shifted sine function. a) y=2cos(t)+3sin(t); b) 7sin(t)-5cos(t).

Exercise) Sanity check the answers you got in problems 3 and 4 by converting the sine functions you derived in problem #3 to a cosine function.

4) a) sinusoidal(t)+cos(t)=sin(t), find sinusoidal(t); b) Verify that your answer is correct.

5.11.3) a) Prove the following identity.

A*cos(wt)+B*sin(wt)=C*cos{wt-c}
-- C=sqrt(A^2+B^2)
-- c is the angle such that cos(c)=A/C and sin(c)=B/C

Put an other way, c=invtan(B/A) or c=invtan(B/A)+180 degrees

The following identity makes it quite clear that any two sinusoidals of the same frequency or wavelength added together is a sinusoidal, ... (of that same wavelength or frequency).

General Sinusoidal Addition Identity (cosine form)

```
A*cos(wt-a)+B*cos(wt-b)=C*cos{wt-c}
  -- C=sqrt(CC^2+SS^2)
  -- c is the angle such that cos(c)=CC/C and sin(c)=SS/C
  -- CC=A*cos(a)+B*cos(b) and SS=A*sin(a)+B*sin(b)

Put an other way, c=invtan(CC/SS) or c=invtan(CC/SS)+180 degrees
```

Proof

```
A*cos(wt-a)+B*cos(wt-b)=

A*cos(wt)cos(a)+A*sin(wt)*sin(a)+
B*cos(wt)cos(b)+B*sin(wt)*sin(b) =

{A*cos(a)+B*cos(b)}cos(wt)+{A*sin(a)+B*sin(b)}sin(wt)=

-----
Substituting A*cos(a)+B*cos(b) into CC
Substituting A*sin(a)+B*sin(b) into SS
-----

CC*cos(wt)+SS*sin(wt)=    {see problem 5
                          {Sinusoidal Addition Problem Set (part 2)
C*cos(wt-c)
-- C=sqrt(CC^2+SS^2)
-- c is the angle such that cos(c)=CC/C and sin(c)=SS/C
-- CC=A*cos(a)+B*cos(b) and SS=A*sin(a)+B*sin(b)

Put an other way, c=invtan(CC/SS) or c=invtan(CC/SS)+180 degrees
```

Proof Complete

Sinusoidal Addition Problem Set (Part 3)

1) Make use of the General Sinusoidal Addition Identity to prove that if any two sinusoidals of the same, frequency or wavelength, are added together, the resultant function is a sinusoidal of the same frequency or wavelength.

2) Express each of the following as a cosine function.
   a) 3cos(t-13°)+4cos(t-25°); b) 7cos(wt-7°)-5cos(wt+5°);
   c) Sanity check the work you did in parts 'a' and 'b', by graphing the functions given vs the functions you calculated. If these have the same graph, you (most likely) did your work correctly.

3) Derive the general sinusoidal addition identity (cosine form)

A*cos(wt-a)+B*cos(wt-b)=C*cos{wt-c}
-- C=sqrt(CC^2+SS^2)
-- c is the angle such that cos(c)=CC/C and sin(c)=SS/C
-- CC=A*cos(a)+B*cos(b) and SS=A*sin(a)+B*sin(b)

4) Derive the general sinusoidal subtraction identity (cosine form).

Note: The equations of ..
the general sinusoidal addition identity (sine form) .. and
the general sinusoidal subtraction identity (sine form)
are composed entirely of sine functions, there are no cosine functions in them.

5) Derive the general sinusoidal addition identity (sine form).

6) Derive the general sinusoidal subtraction identity (sine form).

7) Find a cosine function equal to cos(t)+2cos(t+17°)

8) Find a cosine function equal to 2cos(t-45°)-cos(t+55°)

9) a) Find a sine function equal to 5sin(t-45°)+3sin(t+30°)

10) a) Find a sine function equal to 2sin(t-30°)-3cos(t+45°)

^^^^^^^^^^^^^^^^^^^^^^^^^^

12 Polar: Coordinates, Functions, Graphing

^^^^^^^^^^^^^^^^^^^^^^^^^^

Polar Coordinates

In the study of Coordinate Geometry and Trigonometry, we have made great use of the Cartesian coordinate system. With a coordinate system in place, the location of any point in a plane or space can be specified by a set of coordinates, (an ordered pair or ordered triple of numbers). Entire sets of points, i.e. lines, circles, parabolas etc. can be specified, by equations that give a relationship between the coordinates. The discovery of the Cartesian coordinate system along with Coordinate Geometry by the mathematician and philosopher Descartes in the 1600's is one of the greatest contributions to the study of mathematics. Coordinate Geometry allowed the great power of Algebra to be applied to Geometry problems to an extent never previously possible.

The Cartesian coordinate system is not the only Coordinate system there is, but it was the first one to be discovered and utilized as a place where equations can be used to represent shapes. Another possible coordinate system is the polar coordinate system. In the 2 dimensional polar coordinate system, the first coordinate of a point is the points distance to the origin. The second coordinate is the orientation of the ray which passes through the point and whose end point is the origin. [See definition of orientation in the Angles section]. Example: (1,45°) The distance of this point from the origin is 1. The ray whose end point is at the origin, that passes through this point, has an orientation of 45 degrees.

Unlike in the Cartesian coordinate system, the coordinates of a point in the polar coordinate system are not unique. For example the location of the point (1,45°) could also be specified as (1,405°) where 405°=45°+360°, or as (1,-315°) where -315°=45°-360° or even as (-1,225°) where 225°=45°+180°. In polar coordinates there is an unlimited number of ways to assign coordinates to any one point. A special case is the origin. The origin is specified by putting a zero in the first coordinate as the distance, and the angle or orientation doesn't matter, it can be any angle.

If the polar coordinates of a point are (r,o), the Cartesian coordinates of this point are (x,y) where

I) x=r*cos(o)
II) y=r*sin(o)

The distance from any point (x,y) to the origin is sqrt(x^2+y^2) therefore the polar coordinate r is calculated using the formula

III) r= +/- sqrt(x^2+y^2)

It is left to the student to prove these 3 conversion formulas.

From equations I through III we derive the following. (The student should derive each of the following equations).

$$\text{V) } \cos(o) = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\text{VI) } \sin(o) = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\text{VII) } \tan(o) = \frac{y}{x}$$

Given the Cartesian coordinates of a point, we can make use of any two of equations V, VI or VII to find the second polar coordinate, i.e. the angle o, (0° <= o < 360°). For example, lets find the second polar coordinate of the point (1,1). Making use of equation VII we have tan(o) = y/x = 1/1 = 1. If 1 is entered into a calculator and then the invtan key is pressed, the calculator will return 45 (degrees). The only other (geometrically distinct) angle which has a tangent of 1 is o=45°+180°=225°.

Making use of equation V we have cos(o) = x/sqrt(x^2+y^2) = 1/sqrt(2). If this value is entered into a calculator and then the invcos key pressed, the calculator returns 45 (degrees). The only other (geometrically distinct) angle which has a cosine of 1/sqrt(2) is -45 degrees or 315 degrees. Equation VII tells us o is (45 degrees or 225 degrees). Equation V tells us o is (45 degrees or 315 degrees). 45 degrees is only angle common to both sets of angles, therefore, the second polar coordinate of (1,1) is 45 degrees.

An other way to determine the second polar coordinate of a point expressed in Cartesian coordinates would be to make use of any one of equations V,VI,VII, and the signs of each of the Cartesian coordinates. Making use of any of the equations V,VI,VII will give two possible angles o. One of these angles can be eliminated by by making use of the following table

(+,+)	1st Quadrant	(+,0)	0 degrees
(-,+)	2nd Quadrant	(0,+)	90 degrees
(-,-)	3rd Quadrant	(-,0)	180 degrees
(+,-)	4th Quadrant	(0,-)	270 degrees

For example, lets once again find the second polar coordinate of the point (1,1). Making use of equation VII we have tan(o) = y/x = 1/1 = 1. Making use of the calculator inverse tangent function, we find one possible angle is o=45 degrees. The only other (geometrically distinct) angle which has a tangent of 1 is o+180°=225°. A ray with an orientation of 45 degrees lies in the 1st quadrant. An ray with an orientation of 225 degrees lies in the 3rd quadrant. We note that the signs of the coordinates of the point (1,1) are both positive or (+,+). Therefore the angle o in question has an orientation that lies in the 1st quadrant. Therefore we eliminate o=225 degrees as a possibility and realize the answer we seek is o=45 degrees.

Polar Coordinates Exercises

1) Plot the following points on a sheet of paper. a)(2,45°); b)(3,310°); c)(-2,30°); d)(1,-145°); e)(1,2Pi/3 rads); f)(-3,-2rads); g)(1,1000°); h)(-1,12rads); i)(1,-10,000 rads)

2) Determine the Cartesian coordinates of each of the following points. a)(1,30°); b)(7,232°); c)(0,170°); d)(3,Pi/6 rad); e)(11,3rad); f)(-4,1600°).

3) Determine (using degrees) polar coordinates for the following points. a)(1,1); b)(12,-13); c)(-8,-5); d)(-9,12); e) (0,-3).

4) You know the Cartesian coordinates of a point and wish to find its polar coordinates. You calculated the cosine of the second polar coordinate o, and then made use of the inverse cosine function on your calculator to find that o is possibly the second polar coordinate. What is the other angle that is possibly the second polar coordinate?

5) You know the Cartesian coordinates of a point and wish to find its polar coordinates. You calculated the sine of the second polar coordinate, and then made use of the inverse sine function on your calculator to find that o is possibly the second polar coordinate. What is the other angle that is possibly the second polar coordinate?

6) You know the Cartesian coordinates of a point and wish to find its polar coordinates. You calculated the tangent of the second polar coordinate, and then made use of the inverse tangent function on your calculator to find that o is possibly the second polar coordinate. What is the other angle that is possibly the second polar coordinate?

7) a) Determine the polar coordinates of the point (-7,1). In doing this use equations V and VI to determine the second polar coordinate. b) Do this problem again, this time make use of equation VII and the signs of the Cartesian coordinates to find the second polar coordinate.

Conversion Equations (Preparation for Polar Functions)

We can make use of 4 equations below to convert polar functions to Cartesian functions and visa versa. We have already mentioned and worked with equations I,II and III.

I) x=r*cos(o)
II) y=r*sin(o)

III) r= +/- sqrt(x^2+y^2)
     for example: (1,1) = {sqrt(2),45°} = {-sqrt(2),135°}

IV) o= any angle 'a' where tan(a)=y/x

Exercise: Use equations I and II to prove equations III and IV.

If we take the inverse tangent of both sides of equation IV we have a=invtan(y/x). Since the range of the function invtan is from -90° to 90°, a=invtan(y/x) is the angle (-90°<a<90°) whose tangent is y/x. There is only one such angle. Since (-90°<a<90°), any point (x,y) on the right half of the plane (quadrants 1 and 2) can be represented in polar form as (r,o) where r= sqrt(x^2+y^2) and o = invtan(y/x). Any point (x,y) on the left side of the plane can also be represented in polar form as (r,o) where r= -sqrt(x^2+y^2) and o= invtan(y/x). The proof of this follows.

In the following proof, the following 4 equations hold, (are true).
r= +/- sqrt(x^2+y^2); o=arctan(y/x); (x>0); (y=any number)

(r,o) = {r*cos(o),r*sin(o)} = (x,y) .. {r=sqrt(x^2+y^2}

is representative of any point on the right side of the plane.

If we substitute -r into r we have

(-r,o) = {-r*cos(o),-r*sin(o)} = (-x,-y) .. {r=sqrt(x^2+y^2} ->
(r,o) = {r*cos(o),r*sin(o)} = (-x,-y) .. {r=-sqrt(x^2+y^2}

Because (-x,-y) can represent any point on the left side of the of the plane, (r,o) can also represent any point on the left side of the plane. Therefore any point on the left side of the plane can be represented as (r,o) where r=-sqrt(x^2+y^2) and o=invtan(y/x).

Proof complete

y/x is not defined for x=0. However we can define invtan(y/x) for x=0 if we wish. We do it as follows. If x=0 and y>0, then invtan(y/x)=90°. If x=0 and y<0, then invtan(y/x)=-90°.

Therefore any point (x,y) where x=0, can be written as (r,o) where r= +/- sqrt(x^2+y^2) and o=invtan(y/x).

We have established that every point in the right side of the plane, every point on the left side of the plane, and every point where x=0 can be written as (r,o) where r= +/- sqrt(x^2+y^2) and o=arctan(y/x). Therefore every point on the plane can be represented in polar coordinates as (r,o) where r= +/- sqrt(x^2+y^2) and o= arctan(y/x).

Exercise: Make use of equations I, II, III and IV as needed to express the following points in polar form, where o=invtan(y/x).
a)(1,1); b)(-1,1); c)(-1,-1); d)(1,-1);
e) (1,0); f)(0,1); g)(-1,0); h)(0,-1).

Where x is any number and y is any number, there are many ways we could represent any point (x,y) in polar form. Given that cos(o)=-cos(o+180°) and sin(o)=-sin(o+180°) we have

(r,o)={r*cos(o),r*sin(o)}={r*(-1)cos(o+180°),r*(-1)sin(o+180°)}=
{-r*cos(o+180°),-r*sin(o+180°)}=(-r,o+180°)

Therefore if (r,o) is one to express (x,y) in polar form, then (-r,o+180°) is an other way. Based on this, (r,o+360°*n) and (-r,o+180°+360°*n) n=(... -3, -2, -1, 0, 1, 2, 3 ...) are also other the ways to represent (x,y) in polar form. These last 'two' ways are all of the possible ways to represent (x,y) in polar form.

Exercise: a) Find all polar forms of the point {1,-sqrt(3)}
          b) Find all polar forms of the point {-sqrt(3),1}

Polar Functions

Lets make use of equations

I) x=r*cos(o)
II) y=r*sin(o)

III) r= +/- sqrt(x^2+y^2)
     for example: (1,1) = {sqrt(2),45°} = {-sqrt(2),135°}

IV) o= any angle 'a' where tan(a)=y/x

to convert functions expressed in Cartesian form to polar form and visa versa.

Example .. Express the function y=x in polar form

Method 1

Given y=x

tan(o)=y/x -> tan(o)=x/x -> tan(o)=1 -> o=invtan(1) -> o=45°.
Therefore, o=45° is the line y=x in polar form.

Method 2

y=x -> r*sin(o)=r*cos(o) -> sin(o)=cos(o) -> sin(o)/cos(o)=1 -> tan(o)=1 -> arctan{tan(o)}=arctan(1) -> o=45°. Therefore o=45° is the line y=x in polar form.

Exercise: Does o=45° represent the entire line y=x or just half?
Hint: The entire line, now explain why.

Example .. express o=45° in Cartesian form.

o=45° -> tan(o)=tan(45°) -> sin(o)/cos(o)=1 ->sin(o)=cos(o) -> x/r = y/r -> y=x

note: x=r*cos(o) -> cos(o)= x/r : x=r*sin(o) -> sin(o)= x/r

Polar Functions Exercises

1) a) The Cartesian coordinates of a point are (x,y). a) Give two polar expressions of this point. b) Give all possible polar expressions of this point.

2) A point expressed in polar form is (r,o). Express this point in Cartesian form.

3) A ray has an orientation of o (0°<=o<360°). Assuming the following, what are the possible values of o?. a) tan(o)=2; b) tan(3o)=7; c) tan(o/5)=12.

4) Convert the following to Cartesian form. When you are finished take what you get and convert it back again. Did you end up with what you started with? a) o=30°; b) r=5; c) r=sin(o).

5.12) a) Transform r^2*tan(o)=1 into (y/x)(x^2+y^2)=1
b) Transform y=x/y * (x^2+y^2) into r^2*tan(o)=1,(step by step).

6.12) a) Transform y=1/(2x) into r^2*sin(2o)=1.
b) Transform r^2*sin(2o)=1 into 1/(2x) (step by step).

7) a) Prove r= C/{A*sin(o)+B*cos(o)} is a line.
b) Convert the equation you derived in 'a', to polar form.

8.12) a) Prove r=sin(o)/cos^2(o) is a parabola.
b) Convert the equation you derived in 'a', to polar form.

9) a) Prove r=cos(o) is a circle.
b) Convert the equation you derived in 'a', to polar form.

10) a) Prove r^2*cos^2(o)+r^2*sin^2(o)-2r*cos(o)-2r*sin(o)+1=0 is a circle. b) Convert the equation you derived in 'a', to polar form.

11.12) Make use of polar<->Cartesian conversion formulas, to derive the polar distance formula, i.e. derive the fact that the distance between the two points (r2,o2)&(r1,o1) is sqrt{r2^2+r1^2-2*r2*r1*cos(o2-o1)}.

12) Make use of the law of cosines to derive the polar distance formula, i.e. derive the fact that the distance between the two points (r2,o2)&(r1,o1) is sqrt{r2^2+r1^2-2*r2*r1*cos(o2-o1)}.

13) a) Make use of the Cartesian distance formula to determine the distance between P1(-1,7) and P2(2,-5). b) Determine the polar coordinates of P1 and P2. c) Make use of the polar distance formula to determine the distance between P1 and P2. Did you get the same answer as in 'a'?

14.12) a) Determine where the parabola r=cos(o)/sin^2(o) and the line r=1/{cos(o)-sin(o)} intersect. b) Check your answers by calculating where these functions in Cartesian form intersect.

Graphing Polar Functions

To graph points or functions expressed in polar form, ideally you should use polar graph paper. You can make it yourself or you can buy it. To make it, use a compass to make concentric circles centered at the origin of radius, 1cm, 2cm, 3cm ..., Try to make at least 10 or more concentric circles. Draw them lightly so they won't interfere with your graph.

To graph a polar function, first put the function into the form r=f(o). Then calculate or estimate the range of the function you are about to graph so you can decide what distance scale to use. For example, if you are going to graph r=cos(o) you know the range of of your function is from -1 to 1. Therefore if you have 10 concentric circles, of radius 1cm, 2cm, 3cm ... 10cm that you can make use of. You should scale your distance so that a distance of 1 equals 10 cm.

Then make a table of values of r and o. In this table list a series of angles o. 0°, 10° 30° ... To the side of each of these listings put, (in numeral form) the corresponding f(0°), f(10°), f(20°) ... . If f(0°) is positive, on the ray with an orientation of 0°, mark the point that is a distance of f(0°) from the origin, according to the scale you have chosen. Or if f(0°) is negative, on the ray with an orientation of 0° + 180°, mark the point that is a distance of f(0°) from the origin according to the scale you have chosen. Continue this process for angles beyond 0 degrees, i.e. 10°, 20°, 30° ... until you have completed your graph or until you have plotted all points you wish to plot. As you plot points you may connect them with a smooth curve or wait until you have several points marked and then connect them.

Be aware that graphing the function over a domain span 360 degrees will not necessarily graph an entire function. For example graphing the function r=cos(o) over a domain span of 180 degrees will completely graph the function. r=cos(2o) needs to be graphed over a domain span of 2Pi to graph it completely. r=cos(o/2)needs to be graphed over a domain span of 4Pi to be complete. r=cos{sqrt(2)*o} is not complete for any domain span that is finite.

Graphing Polar Functions Problem Set

1) a) Graph r=sin(o) using polar graphing techniques. b) Check your answer by comparing the graph of the function with the Cartesian equation of the function.

2) a) Convert y=x-1 to polar form. b) Graph the function you derived in part 'a' using polar graphing techniques. c) Check your answer by comparing the graph of the function with the Cartesian equation of the function.

3) a) Convert y=(x^2)/6 to polar form. b) Graph the function you derived in part 'a' using polar graphing techniques. c) Check your answer by comparing the graph of the function with the Cartesian equation of the function. c) Replace o with o+45° in the function you derived in part 'a'. Graph this new function.

4) Graph r=cos(o), r=sin(o), r=-cos(o), and r=-sin(o) in the same graph space to see an interesting pattern. Or if you want to save time, convert each of these equations to Cartesian form. Can you see the pattern the graphs of these functions would make by looking at their Cartesian equations?

5) The following functions are known as cartoids. Graph any or all. a) r=1+cos(o); b) r=1+sin(o); c) r=1-cos(o); d) r=1-sin(o); e) Check you answers by graphing these functions using a graphing calculator or computer. f) Using a graphing calculator or computer graph all 4 cartiods in the same graph space to see an interesting pattern.

6) a) r=o is a double spiral. a) Graph the spiral r=o (r>0); b) Graph the spiral r=o (r<0).

^^^^^^^^^^^^^^^^^^^^^^^^^^

13 de Moivre's Formula and Complex Numbers

^^^^^^^^^^^^^^^^^^^^^^^^^^

Complex Numbers

The roots of the 2nd degree polynomial (1/2)x^2+3x+8=0 are -3+sqrt(-7) and -3-sqrt(-7). If either of these values are substituted into this second degree polynomial, the result will be 0. The value sqrt(-7) is an referred to as an imaginary number. This name probably originated, because when such numbers were discovered, they couldn't be found anywhere on the number line, nor could their value be comprehended at the time, therefore they were thought to be imaginary,(i.e. not real). The square root of any negative number is an imaginary number. By definition, a complex number is a real number added to an imaginary number. By definition then, imaginary numbers are complex numbers, because any imaginary i, is 0+i, (0 is a real number).

Complex numbers were probably first discovered as the roots of certain quadratic equations. Complex numbers are extremely useful in physics and electrical engineering. A certain distance multiplied by 1/2 is half the given distance. Likewise a certain distance multiplied by 2 is twice the given distance. There are equations in physics (relativity) which suggest that any distance multiplied by an imaginary number is a certain amount of time, i.e.
300 million meters * sqrt(-1) = 1 second. (The speed of light is 300 million meters per second) Electric fields and magnetic fields have a similar relationship as time and distance. An electric field multiplied by an imaginary number is an electric field and visa versa. If two events are separated by distance only, their separation can be expressed as A meters, where A is a real number. If two events are separated by time only but not distance their separation can be expressed as A meters where A is an imaginary number. If two events are separated by time and distance, their separation can be expressed as A meters where A is a complex number. Clearly imaginary numbers have a magnitude, however their magnitude is not of the same type as the magnitude of real numbers.

Just as 1 is the basic unit of the real numbers, the imaginary number sqrt(-1) is the basic unit of the imaginary numbers. This quantity is given the symbol designation of i, i=sqrt(-1).
i does not lean towards being positive or negative. Since i has a magnitude but it doesn't lean towards being positive or negative, if we were to plot its position with respect to the number 0, it would have to be away from zero, but it would not be "closer to" the positive numbers than the negative numbers or visa versa. Just as the number line can be used to represent the set of real numbers,

a number plane can be used to represent the set of complex numbers. The number plane includes the number line. In the number plane, the positive x axis represents the positive numbers. The origin represents 0. The negative x axis represents the negative numbers. The point (0,1) represents the number i or sqrt(-1). This is a good position on the number plane for i, because it is away from 0, one unit away, and (0,1) is not closer to the negative numbers than the positive numbers or visa versa. 3i is at the point (0,3), -5i is at the point (0,-5). The complex number 2+7i is at the point (2,7) etc. In general, where a and b are real numbers, the complex number a+bi is located at the point (a,b) on the number plane, a is the real part of this complex number and bi is the imaginary part. The complex number a+bi (standard form) can also be written as (a,b) (Cartesian form). It was previously noted that imaginary numbers are complex numbers. Real numbers are also complex numbers, because any real number a can be represented as a+bi, (a and b are real numbers) where b is 0. Just as we have a number line, we have a number plane, and all numbers on the number plane are complex numbers and can be represented as a+bi, where a and b are real numbers.

Two complex numbers are added as follows

a+bi + c+di = (a+c)+(b+d)i {Standard form of complex numbers

or

(a,b) + (c,d) = (a+c,b+d) {Cartesian form of complex numbers

Two complex numbers are multiplied as follows. {Keep in mind that i= sqrt(-1), therefore i*i= -1. Also keep in mind, complex numbers are commutative and the distributive property also applies.

(a+bi)*(c+di) = a(c+di)+bi(c+di) = ac+adi+bic+bidi =
ac+adi+bci+bdii = ac-bd +(ad+bc)i

-) Perform the following calculations. a) 5+(2-i); b)(1+i)-(-3+5i); c) -(-1,8)+(2,4).

-) Perform the following calculations. a) 3*(1+2i); b)(4-6i)/2; c) (2-3i)(1+7i); d) (1+2i)(5+4i).

Adding complex numbers is similar to adding vectors. Multiplying complex numbers is similar to multiplying monomials, can you see why this is so?

-) a) Add the following complex numbers together. b) Multiply the following complex numbers together. I) h+ki, m+ni; II) a+bi, c-di

A useful concept when dealing with complex numbers is the conjugate of a complex number, the conjugate of a complex number a+bi is a-bi. Any complex number multiplied by its conjugate is a real number.

Prove this for yourself now. Where A and B are complex numbers, A/B is also a complex number. Remember a complex number is any number that can be written as a+bi where a and b are real and i is sqrt(-1). It isn't obvious that (A+Bi)/(C+Di) is a complex number or even how to perform this division. Enough information has been given in this discussion to enable you do this. Try to determine for yourself how to perform this division and then prove that the quotient of two complex numbers is a complex number. Try to do this for yourself before looking at the proof below.

Theorem: The quotient of two complex numbers is a complex number.

Proof

```
a+bi   a+bi   c-di   ac-cdi+bci+bd   (ac+bd)+(bc-cd)i
---- = ---- * ---- = ------------- = ---------------- =
c+di   c+di   c-di   cc-cdi+cdi+dd        cc+dd


ac+bd     bc-cd
----- + i -----   <---  This is a complex number, therefore ...
cc+dd     cc+dd
```

Proof Complete

If you were not able to prove that the quotient of two complex numbers is a complex number by yourself before looking at the above proof. Do it now, without looking at the proof above. The following rules all apply to complex numbers just as they do for real numbers.

Where A and B are complex numbers.

```
Closure: A+B, A-B, A*B, A/B (B!=0) ... are complex numbers

Associative A+B+C = (A+B)+C = A+(B+C)
            A*B*C = (A*B)*C = A*(B*C)

Commutative A+B=B+A
            A*B=B*A

Distributive A(B+C)=AB+AC

Identity Element A+0=A  0 is the additive identity element
                 A*1=A  1 is the multiplicative identity element
```

There exists a complex number 0 (zero), such that when 0 is added to ANY complex number A, the sum of 0 and A is A. Zero is the additive identity element of the complex numbers.

There exists a complex number 1 (one), such that when ANY complex number A is multiplied by 1, the product of 1 and A is A. 1 is the multiplicative identity element of the complex numbers.

```
Inverse Element A + -A  = 0   0 is additive identity element
                A * 1/A = 1   1 is multiplicative identity element
```

An inverse element provides a means of returning to the identity element.

Every complex number A has a (unique) additive inverse element -A, (-A is also a complex number), such that A + -A = 0.

Every complex number A (A!=0) has a (unique) multiplicative inverse element 1/A (1/A is also a complex number) such that A * 1/A = 1.

Addition and multiplication are operators. So too are subtraction and division. The operators of subtraction and division are defined below.

```
Definition of Subtraction  A-B = A + -B

Definition of Division   A/B = A * 1/B    (B!=0)
```

1.13.2) Without looking, at anything else, do the following problems relating to real and complex numbers.
  a) Illustrate and explain closure for addition and multiplication.
  b) Illustrate associativity for addition and multiplication.
  c) Illustrate commutativity for addition and multiplication.
  d) Illustrate the distributive property.
  e) Illustrate and explain identity element for addition and multiplication.
  f) Illustrate and explain inverse element for addition and multiplication.
  g) Define subtraction and division.

2.13.2) Its given that a(b+c)=ab+ac, prove (b+c)a=ab+ac

Complex Number Problem Set

1) Simplify and plot (on the number plane) the following. a) i^1 ; b) i^2; c) i^3; d) i^4; e) i^5; f) i^3007; g) i^5432; h) i^11142.

2) A=3+2i, B=5+4i, C=(2,-7), D=(m,n), E=p+qi, F=2, G=(r,-s), H=i
Perform the following calculations, and leave the answers in standard complex form, i.e. as a real number added to an imaginary number. a) A+B; b) A-B; c) F*A; d) A*B; e) A/B; f) 1/A) g) A+C; h) B-C; i) 1/C; j) D*E; k) D/E; l) B/C; m) B*C; n) B/C; o) C/B; p) D/G; q) G/D; r) 1/D; s) 1/G; t) 1/B; u) 1/H; v) C*H; w) C/H

Polar Representation of Complex Numbers

The similarities between 2 dimensional polar coordinates and polar representation of complex numbers are strong. The student should study the section Polar Coordinates in this book before studying this section.

Given that any complex number is represented by a particular point on the number plane, it is natural to wonder if complex numbers like points on a plane, can be represented using polar coordinates. Yes they can and are. For example, the number complex number 3+4i or (3,4) is written in polar form as {sqrt(3^2+4^2),invtan(4/3)} or {5,53.13°).

From here on in this section, we will not distinguish between a complex number and the point on the complex plane that represents it When either one is written or mentioned, think of both.

(x,y) is a complex number expressed in Cartesian form. (r,o) and r<o (rho angle theta) are complex numbers expressed in polar form. The first polar coordinate 'r' is the magnitude of the complex number. The magnitude of a complex number is the distance from the origin to the complex number. The second polar coordinate of a complex number 'o' is the argument of the complex number. The argument of a complex number, is the orientation of the ray whose endpoint is the origin, that passes through the complex number. 'o' is an angle.

-) What is the magnitude and argument of each of the following complex numbers? a) (1,2); b) (1,7°); c) (r,o)

See if you can do the following problems before being shown how.
A discussion of how to do these problems follows these problems.

-) Convert the following complex numbers to polar form. a) (1,1); b) -3+i; c) {-1,-sqrt(3)}; d) 2-3i

-) Convert the following complex numbers to Cartesian form. a)(1,45°); b) 2<120°; c)(3,240°); d)(4,300°); e) -1<45°

Where (r,o) is the polar form of a complex number we use equations I and II below to find the Cartesian coordinates (x,y) of the complex number.

I) x=r*cos(o)
II) y=r*sin(o)

We derive the following equations III, IV, V and VI because they help provide a way find the polar coordinates of a complex number, if the if the Cartesian form is given.

The distance formula is used to derive equation III.

III) $r=\sqrt{x^2+y^2}$

From equations I through III we derive equations IV, V and VI that follow.

$$\text{IV) } \cos(o)= \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\text{V) } \sin(o)= \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\text{VI) } \tan(o)= \frac{y}{x}$$

Following is a way to convert a complex number from Cartesian form to polar form.

Example: Convert (-1,1) to polar form.

To find the magnitude use equation III) $r=\sqrt{x^2+y^2}$.

$r= \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

To find the magnitude, we can make use of any one of equations IV, V or VI. We choose to make use of equation IV)

$$\text{IV) } \cos(o)= \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

Taking the arccos of both sides of equation IV we get

$o = \arccos\{-1/\sqrt{2}\}$ or $o = 135°$

Making use of this answer and our knowledge of trigonometry we see the possible angles 'o' can equal are 135° or -135°. To decide which of these angles is the correct one, we note the point (-1,1) is in the 2nd quadrant, therefore o=135°.

Therefore (-1,1) in polar form is $\{\sqrt{2},135°)$

-) Make use of the method just shown, to convert $\left[\frac{\sqrt{2}}{2},\frac{1}{2}\right]$ to polar form.

Forms of a Complex Number

a+bi	3-4i	... standard form	(a is the x coordinate)
(a,b)	(7,2)	... Cartesian Form	(b is the y coordinate)
(r,o)	(5,12°)	... polar form	(r is the magnitude)
r<o	5<12°	... polar form	(o is argument)
			< means angle

Polar Representation of Complex Numbers Problem Set

1) Convert the following complex numbers to Cartesian form.
   a)(1,45°); b)(2,30°); c) 12<210°; d)(-6,-18°);
   e)(4,12rads); f) (r,o)

2) Convert the following complex numbers to polar form. a) 1+2i;
   b)-3+4i; c)(-5,-6); d)(7,-8); e) a+bi

3) a) Convert the complex number cos(o)+sin(o)i to polar form.
   b) Convert the complex number A*cos(o)+A*sin(o)i to polar form.

4) The Complex number (r,o) is in polar form. Convert this complex number to Cartesian form and then to standard form. (assume o is in the 1st quadrant).

5) a) Calculate each of the following and leave the answers in polar form. a)(3,32°)+(8,237°); b)(2,0°)+(3,30°); c) 3*(4,32°)

6) a) Calculate each of the following and leave the answers in polar form. a)(3,32°)*(8,237°); b)(2,0°)*(3,30°); c) 3*(4,32°)

Theorem Preceding de Moivre's Formula

de Moivre's Formula, and the unnamed theorem that is its basis are unexpected jewels of trigonometry. The unnamed theorem proceeding de Moivre's formula states

If A' and B' are complex numbers, where A'=A<a and B'=B<b, then
magnitude of A'*B' = (magnitude of A') * (magnitude of B') and
argument of A'*B' = (argument of A') + (argument of B')
i.e. A<a * B<b = (A*B)<(a+b) . . . {where < means angle}

For example, if A=(2,13°) and B=(3,34°). Then A*B=(2*3,13°+34°) = (6,47°). Another example, if H is (4,-16°) and K is (-5,100°) then H*K = (4*-5,-16°+100°) = (-20,84°) = (20,264°) = (20,-96°).

The proof of the theorem which is the basis of de Moivre's formula follows.

A' * B' =   A'and B' are complex numbers
A<a * B<b =   expressing this in Cartesian form we have

{A*cos(a)+A*sin(a)i} * {B*cos(b)+B*sin(b)i}=
A*{cos(a)+sin(a)i} * B*{cos(b)+sin(b)i}=
A*B*{cos(a)+sin(a)i}*{cos(b)+sin(b)i}=
A*B*[cos(a)cos(b)-sin(a)sin(b)+{cos(a)sin(b)+sin(a)cos(b)}i]=

making use of the cosine and the sine addition formulas we get

(A*B)*{cos(a+b) + sin(a+b)i} =

expressing this in polar form we have

(A*B)<(a+b) or (A*B,a+b)

Proof Complete

Theorem Preceding de Moivre's Formula Problem Set

It should be obvious at this point that adding complex numbers is easier if they are in Cartesian form and multiplying complex numbers can be easier if they are in polar form.

1) a) Calculate (1,45°) * (2,30°) without making use of the theorem that precedes de-Moivres formula. b) Calculate (1,45°) * (2,30°) by making use of the theorem that precedes de-Moivres formula. c) Verify that the answers you got in parts 'a' and 'b' are the same.

2) Prove: If two complex numbers lie in the unit circle centered at the origin, then their product also lies on the unit circle centered at the origin.

3) Prove the theorem that precedes de Moivre's Formula.

de Moivre's Formula

We do not prove that de-Moivres formula is true. Rather we provide a motivation to see why it is true.

Where (r,o) is a complex number expressed in polar form, we have in accordance with the previous theorem

(r,o)^1=(r,o)
(r,o)^2=(r,o)*(r*o)=(r^2,2o)
(r,o)^3=(r,o)*(r*o)*(r,o)=(r^3,3o) etc.
(r,o)^n = (r^n,n*o) <--- de Moivre's formula
This motivation allows one to see that deMoivre's formula is true

for exponents that are positive integers. However de-Moivre's formula is also true for all real (integer, fractional and irrational) exponents n (-infinity < n < infinity).

de Moivre's Formula Problem Set

2) Provide a motivation showing that de Moivre's formula is true.

4) Calculate and then plot the following points on the number plane. a) {1,15°}^n for n= 0,1,2,3,4,5,6,7;
b) {sqrt(2),15°}^n for n= 0,1,2,3,4,5,6,7.

Using de Moivres formula, it can be demonstrated (proved) that every number has; 2 square roots, 3 cube roots, 4 4th roots etc, i.e. every number has n nth roots for n = {2,3,4 ... .

For example suppose we wish to find all cube roots of 1. We already know that 1*1*1=1, therefore 1 is a cube root of 1. According to de Moivres formula, (1,120°)*(1,120°)*(1,120°)=(1,360°)= 1. Therefore (1,120°) is also a cube root of 1.

The other cube root of 1 is (1,2*120°) or (1,240°), can you see why this is also a cube root of 1?. Making use of de Moivres formula we have (1,240°)^3=(1,2*120°)^3 = (1,2*360°) = 1.

Therefore the 3 cube roots of 1 are 1, (1,120°) and (1,240°).

1) In the previous example, we determined that (1,120°) is a cube root of 1. a) Convert (1,120°) to standard form; b) Multiply the number you calculated in part 'a' by itself 3 times to verify it is a cube root of 1.

2) a) Calculate the 4 4th roots of 1; b) Convert each of these 4th roots to standard form; c) Verify that the numbers you calculated in part 'b' are indeed 4th roots of 1.

3) a) Determine the 3 cube roots of 27
b) Determine the 5 5th roots of -32
c) Determine the 2 square roots of 1+i

4) All complex numbers, zero excepted, have (exactly) n, nth roots where n = {1,2,3 ...). Give an explanation why (prove if possible) that all positive numbers have (exactly) n, nth roots. b) Give an explanation why or prove if possible that all complex numbers have (exactly) n, nth roots. Note: Zero is the one exception.

^^^^^^^^^^^^^^^^^^^^^^^^^^

14 Trigonometry Expression Manipulation
Trigonometry Equation Solving

^^^^^^^^^^^^^^^^^^^^^^^^^^

Throughout this book, one of the skills the student has been learning, is to manipulate trigonometric expressions. For example, the student by now should be able to express tan(x) in terms of cos(x) and visa versa. This section focuses on learning how to manipulate and also simplify trigonometry expressions. There are problems the student will encounter where increasing this skill will be beneficial.

In this chapter several example problems are given. The best way to study these example problems is to try to solve them on your own first without looking at the solution.

If you are not able to do one of the problems after struggling with it, mark that problem then go on to do the problems in the following pages. After you have completed this section, or later in the section, go back and try to do the marked problems you were unable to do the first time through. In this chapter, sometimes the studying the examples given and doing the problems given after a certain problem will give you the skills and insights needed to do that certain problem.

Example Problem

Express sin(x) in terms of cos(x)

Solution

sin^2(t)+cos^2(t)=1 -> sin^2(t)=1-cos^2(t) ->
sin(t)= +/- sqrt{1-cos^2(t)} <-- The answer.

1) Make a table similar to the one below. Replace each number in the table with function(s) that you calculate. All the functions of a given row are to be equal, but they are to be expressed differently. Express each function in terms of the function that is at the top of its column. For example, replace the number 2 with a function equal to sin(x), since sin(x) is on this row, but express this function in terms of cos(x), since cos(x) is at the top of this column. .. Replace the number 2 with +/- sqrt{1-cos^2(t)}. Compare your answers with those below.

	sin(t)	cos(t)	tan(t)	csc(t)	sec(t)	cot(t)
sin(t)	1	2	3	4	5	6
cos(t)	7	8	9	10	11	12
tan(t)	13	14	15	16	17	18
csc(t)	19	20	21	22	23	24
sec(t)	25	26	27	28	29	30
cot(t)	31	32	33	34	35	36

1) sin(t)
2) +/- sqrt{1-cos^2(t)}
3) tan(t)/sqrt{1+tan^2(t)}
4) 1/csc(t)
5) +/- sqrt{sec^2(t)-1}/sec(t)
6) +/- 1/sqrt{1+cot^2(t)}
7) +/- sqrt{1-sin^2(t)}
8) cos(t)
9) +/- 1/sqrt{1+tan^2(t)}
10) +/- sqrt{csc^2(t)-1}/csc(t)
11) 1/sec(t)
12) +/- 1/sqrt{1+cot^2(t)}
13) +/- sin(t)/sqrt{1-sin^2(t)}
14) +/- sqrt{1-cos^2(t)}/cos(t)
15) tan(t)
16) +/- 1/sqrt{csc^2(t)-1}
17) +/- sqrt{sec^2(t)-1}
18) 1/cot(t)
19) 1/sin(t)
20) +/- 1/sqrt{1-cos^2(t)}
21) +/- sqrt{1+tan^2(t)}/tan(t)
22) csc(t)
23) +/- sec(t)/sqrt{sec^2(t)-1}
24) +/- sqrt{1+cot^2(t)}
25) +/- 1/sqrt{1-sin^2(t)}
26) 1/cos(t)
27) +/- sqrt{1+tan^2(t)}
28) +/- csc(t)/sqrt{csc^2(t)-1}
29) sec(t)
30) +/- sqrt{1+cot^2(t)}/cot(t)
31) +/- sqrt{1-sin^2(t)}/sin(t)
32) +/- cos(t)/sqrt{1-cos^2(t)}
33) 1/tan(t)
34) +/- sqrt{csc^2(t)-1}
35) +/- 1/sqrt{sec^2(t)-1}
36) cot(t)

Do the following problem without referring to the previous problem or its answers. Doing this problem independently will help to provide the extra practice needed to make this material second nature.

2) Do this problem without referring to the previous problem. Assume that 'a' equals sin(x) of the first column, then go on to calculate what each of the functions in the second column are in terms of 'a'. Once you have done this, do the same for each of the other functions in the first column. Two examples: if a=sin(x), then sin(x)=a. If a=sin(x). cos(x) would be calculated as follows. sin^2(x)+cos^2(x)=1 -> a^2+cos^2(x)=1 -> cos^2(x)=1-a^2 -> cos(x)= +/- sqrt(1-a^2).

given that a =	sin(x)	find	sin(x)	in terms of a.
	cos(x)		cos(x)	
	tan(x)		tan(x)	
	sec(x)		sec(x)	
	csc(x)		csc(x)	
	cot(x)		cot(x)	

Example Problem

Simplify the term cos{invsin(c)}, such that the simplified form does not contain any trigonometry functions. Solution, the term invsin(c) is verbalized as the angle whose sine is c. The problem is asking what the cosine of this angle is. Let a=invsin(c). Therefore we have sin^(a)+cos^(a)=1 -> c^2+cos^2(a)=1 -> cos(a)= +/- sqrt(1-c^2). Therefore cos{invsin(c)}= +/- sqrt(1-c^2).

The following problem was given in the "Inverse Trigonometry Functions" section. If you didn't these problems before or didn't do them well, do them now. In doing the problems in problem 3, do not refer to the previous two problems or other problems in problem 3. Doing these independently will help to provide the practice needed to make this material second nature.

3) Simplify the following expressions, the simplified forms will not contain any trigonometry functions. Hint: Upon seeing an inverse function, invcos(t) for example, remember to think the words. "The" angle whose cosine is t. (This problem was given in the Inverse Trigonometry Functions section. If you didn't do it then, do it now).

a) cos{invcos(c)}; b) cos{invsin(c)}; c) cos{invtan(c)};
d) cos{invsec(c)}; e) cos{invcsc(c)}; f) cos{invtan(c)};

g) sin{invcos(c)}; h) sin{invsin(c)}; i) sin{invtan(c)};
j) sin{invsec(c)}; k) sin(invcsc(c)}; l) sin(invtan(c)};

m) tan{invcos(c)}; n) tan{invsin(c)}; o) tan{invtan(c)};
p) tan{invsec(c)}; q) tan{invcsc(c)}; r) tan{invcot(c)};

s) sec{invsin(c)}; t) sec{invcos(c)}; u) sec{invtan(c)};
v) sec{invsec(c)}; w) sec{invcsc(c)}; x) sec{invcot(c)};

y) csc{invsin(c)}; z) csc{invcos(t)}; aa) csc{invtan(c)};
ab) csc{invsec(c)}; ac) csc{invcsc(t)}; ad) csc(invcot(c)};

ae) cot{invsin(c)}; af) cot{invcos(c)}; ag) cot{invtan(c)};
ah) cot{invcsc(c)}; ai) cot{invsec(c)}; aj) cot{invcot(c)}.

4) Write these trigonometric expressions as algebraic expressions.
a) sin{invsin(x)+invcos(x)}; b) cos{invtan(1)+arccos(x)}

5) Assuming sec(u)= -3/2 and tan(u)> 0, find all 6 trigonometry functions. Answers are below. Calculate answers before looking at them. STOP

sin(u)= -sqrt(5)/3 csc(u)= 3*sqrt(5)/5
cos(u)= -2/3 sec(u)= -3/2
tan(u)= sqrt(5)/2 cot(u)= 2*sqrt(5)/5

(For the rest of this section, assume angles are given in radians)

If you need to know if an identity is true and you are unable to prove it analytically, (using techniques of algebra) it is possible to prove it with a high degree of confidence within an interval using graphing. For example to "prove" sin(2x)=2cos(x)sin(x), graph the function y=sin(2x)-2cos(x)sin(x) over the interval 2Pi. If the graph is 0 over this entire interval, you have "proved" the identity. In this case the interval [0,2pi] was chosen as the interval to graph the function because the cos(x) and sin(x) have the longest periods and these periods are 2pi. It is up to the student to chose an appropriate interval to graph the test function.

Not every example problems in the rest of this chapter teaches students everything necessary to do the problems that are associated with them. In addition to what is learned in the example problems, students will have to make use of their own creativity, information previously learned in this book or even example problems further ahead in the chapter.

If you can't do a particular problem, mark that problem as one that couldn't be done. Then continue on with the other problems in the section. When you reach the end of the section, go back and attempt to do the problems you couldn't do the first time through. By this time you will have gained more problem solving experience and may learned what is necessary to do the problems you couldn't do the first time through.

Example 1

Prove sin(x)-cos^2(x)sin(x) =sin^3(x)

Proof

sin(x)-cos^2(t)sin(t)=
sin(t){1-cos^2(t)}=
sin(t)sin^2(t)=
sin^3(t)

Proof Complete

Example Problem 1.1

Prove   1+tan^2(t)=sec^2(t) [This is one of the main identities]

Proof

1+tan^2(t)=
1+sin^2(t)/cos^2(t)=
cos^2(t)/cos^2(t) + sin^2(t)/cos^2(t)=
{cos^2(t)+sin^2(t)}/cos^2(t)=
1/cos^2(t)=
sec^2(t)

Proof Complete

a) Prove 1+cot^2(t)=csc^2(t) [This is one of the main identities]
b) Prove sin(t){csc(t)-sin(t)}=cos^2(t)
c) Prove sin(pi/2-t)csc(t)=cot(t)
d) Prove cot(pi/2-t)cos(t)=sin(t)
e) Prove cos^2(t)/{1-sin(t)}=1+sin(t)
f) Prove 1/{tan^2(t)+1}=cos^2(t)

Example 2

$$\text{Prove} \quad \frac{\sin(t)}{1+\cos(t)} + \frac{\cos(t)}{\sin(t)} = \csc(t)$$

Proof

$$\frac{\sin(t)}{1+\cos(t)} + \frac{\cos(t)}{\sin(t)} = \frac{\sin(t)\sin(t) + \cos(t)\{1+\cos(t)\}}{\{1+\cos(t)\}\sin(t)} =$$

$$\frac{\sin^2(t)+\cos^2(t)+ \cos(t)}{\{1+\cos(t)\}\sin(t)} = \frac{1+\cos(t)}{\{1+\cos(t)\}\sin(t)} = \frac{1}{\sin(t)} = \csc(t)$$

Proof Complete

a) Prove $\csc(o)\tan(o)=\sec(o)$
b) Prove $\cos(o)\sec(o)-\cos^2(o)=\sin^2(o)$
c) Prove $1-\sin^2(o)/\{1-\cos(o)\}=-\cos(o)$
d) Prove $\tan(o)/\{1+\sec(o)\}+\{1+\sec(o)\}/\tan(o)=2\csc(o)$
e) Prove $\csc(\pi/2-o)/\tan(-o)=-\csc(o)$
f) Do Example 2 without looking at the provided solution

Example 3

factor $\csc^2(t)-\cot(t)-3$

Solution

Use the identity $\csc^2(t)=1+\cot^2(t)$ to rewrite the expression in terms of cotangent.

$\csc^2(t)-\cot(t)-3=$
$\{1+\cot^2(t)\}-\cot(t)-3=$
$\cot^2(t)-\cot(t)-2=$
$\{\cot(t)-2\}\{\cot(t)+1\}$

Factor and then where possible simplify the following expressions.

a) $\cot^2(t)-\cot^2(t)\cos^2(t)$
b) $\{\cos^2(t)-4\}/\{\cos(t)-2\}$
c) $\tan^4(t)+2\tan^2(t)+1$
d) $\sec^4(t)-\tan^4(t)$
e) $\csc^3(t)-\csc^2(t)-\csc(t)+1$
f) Do Example 3 without looking at the provided solution

Example 4

Factor this trigonometric expression

csc^2(x)-cot(x)-3

solution

csc^2(x)-cot(x)-3=
{1+cot^2(x)}-cot(x)-3=
cot^2(x)-cot(x)-2=
{cot(x)-2}{cot(x)+1}

Factor and then simplify the following.

a) sin^4(x)-cos^4(x)
b) Do Example 4 without looking at the provided solution

Example 5

simplify sin(t)+cot(t)cos(t)

Solution

First, rewrite cot(t) in terms of sine and cosine

sin(t)+cot(t)cos(t)=

```
        cos(t)
sin(t)+ ------ cos(t)=
        sin(t)

sin^2(t)+cos^2(t)
-----------------=
     sin(t)

  1
------ =
sin(t)
```

csc(t)

Simplify each of the following

a) tan(x)-sec^2(x)/tan(x)
b) 1/{1+cos(x)} + 1/{1-cos(x)}
c) 1/{sec(x)+1} - 1/{sec(x)-1}
d) cos(x)/{1+sin(x)} + {1+sin(x)}/cos(x)
e) Do Example 5 without looking at the provided solution

Example 6

Rewrite 1/{1+sin(x)} so it is not in fractional form.

Solution

$$\frac{1}{1+\sin(x)} =$$

$$\frac{1}{1+\sin(x)} \, \frac{1-\sin(x)}{1-\sin(x)} =$$

$$\frac{1-\sin(x)}{1-\sin^2(x)} =$$

$$\frac{1-\sin(x)}{\cos^2(x)} =$$

$$\frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos^2(x)}$$

sec^2(x)-sec^2(x)sin(x)=

this is an optional step

sec^2(x){1-sin(x)}

Put the following into non fractional form

a) sin^2(t)/{1-cos(t)}
b) 5/{tan(x)+sec(x)}
c) 3/{sec(x)-tan(x)}
d) tan^2(x)/{1+csc(x)}
e) Do Example 6 without looking at the provided solution

Example 7

Prove $\{\sec^2(o)-1\}/\sec^2(o)=\sin^2(o)$

Proof

$$\frac{\sec^2(t)-1}{\sec^2(o)} =$$

$$\frac{\{\tan^2(o)+1\}-1}{\sec^2(o)} =$$

$$\frac{\tan^2(o)}{\sec^2(o)} =$$

$$\tan^2(o)\cos^2(o) =$$

$$\frac{\sin^2(o)}{\cos^2(o)}\cos^2(o) =$$

$$\sin^2(o)$$

Proof Complete

Another method of solving the same problem

$$\frac{\sec^2(o)-1}{\sec^2(o)} = \frac{\sec^2(o)}{\sec^2(o)} - \frac{1}{\sec^2(o)} = 1-\cos^2(o)= \sin^2(o)$$

prove the following

a) $\cos^2(B)-\sin^2(B)=1-2\sin^2(B)$
b) $\tan^2(o)+5=\sec^2(o)+4$
c) $\{1+\sin(x)\}\{1-\sin(x)\}=\cos^2(x)$
d) $\cot^2(y)\{\sec^2(y)-1\}=1$
e) Do Example 7 without looking at the provided solution

Example 8

Prove

1/{1-sin(o)} + 1/{1+sin(o)} = 2sec^2(o)

Proof

```
   1          1
-------- + -------- =
1-sin(o)   1+sin(o)


1+sin(o) + 1-sin(o)
-------------------- =
{1-sin(o)}{1+sin(o)}


    2
---------- =
1-sin^2(o)


   2
-------- =
cos^2(o)


2sec^2(o)
```

Proof Complete

Prove the following

a) {cos(x)-cos(y)}/{sin(x)+sin(y)}+{sin(x)-sin(y)}/{cos(x)+cos(y)}=0
b) {tan^2(x)+1}(cos^2-1) = tan^2(x)
c) 2sec^2(x)-2sec^2(x)sin^2(x)-sin^2(x)-cos^2(x)=1
d) csc(x){csc(x)-sin(x)}+{sin(x)-cos(x)}/sin(x)+cot(x)=csc^2(x)
e) Do Example 8 without looking at the provided solution

Example 9

Prove $\tan(x)+\cot(x) = \sec(x)\csc(x)$

This is a difficult problem, unless functions are decomposed into sine and cosine.

Proof

$\tan(x)+\cot(x) =$

$$\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} =$$

$$\frac{1}{\cos(x)\sin(x)} =$$

$$\frac{1}{\cos(x)} * \frac{1}{\sin(x)} =$$

$\sec(x)\csc(x)$

Proof Complete

Prove the following

a) $\tan(x)\cot(x)/\cos(x)=\sec(x)$
b) $\{1+\csc(o)\}/\sec(o)-\cot(o)=\cos(o);$
c) $\sin(x)[1-2\cos^2(x)+\cos^4(x)]=\sin^5(x)$
d) $\csc^4(o)- \cot^4(o)=2\csc^2(o)-1$
e) Do Example 9 without looking at the provided solution

Example 10

Prove sec(x)+tan(x)=cos(x)/{1-sin(x)}

Proof

```
 cos(x)
-------- =
1-sin(x)


cos(x)    1+sin(x)
-------- --------- =
1-sin(x)  1+sin(x)


cos(x)+cos(x)sin(x)
------------------- =
    1-sin^2(x)


cos(x)+cos(x)sin(x)
------------------- =
     cos^2(x)


 cos(x)    cos(x)sin(x)
-------- + ------------ =
cos^2(x)     cos^2(x)


  1     sin(x)
----- + ------ =
cos(x)  cos(x)


sec(x)+tan(x)
```

Proof Complete

Prove the following

a) sin(b)/{1-cos(b)}={1+cos(b)}/sin(b)
b) cot(a)/{csc(a)-1}={csc(a)+1}/cot(a)
c) Do Example 10 without looking at the provided solution

Example 11

Prove $\cot^2(o)/\{1+\csc(o)\} = \{1-\sin(o)\}/\sin(o)$

We will simplify both sides and see if we can arrive at a common place.

Proof

$$\frac{\cot^2(o)}{1+\csc(o)} = \frac{\csc^2(o)-1}{1+\csc(o)} = \frac{\{\csc(o)-1\}\{\csc(o)+1\}}{1+\csc(o)} = \csc(o)-1$$

$$\frac{1-\sin(o)}{\sin(o)} = \frac{1}{\sin(o)} - \frac{\sin(o)}{\sin(o)} = \csc(o)-1$$

Proof Complete

Prove the following

a) $\{\tan^3(a)-1\}/\{\tan(a)-1\}=\tan^2(a)+\tan(a)+1$
b) $\{\sin^3(b)+\cos^3(b)\}/\{\sin(b)+\cos(b)\}=1-\sin(b)\cos(b)$
c) Do Example 11 without looking at the provided solution

Example 12

Prove tan^4(x)=tan^2(x)sec^2(x)-tan^2(x)

Proof

tan^4(x)=
tan^2(x)tan^2(x)=
tan^2(x){sec^2(x)-1}=
tan^2(x)sec^2(x)-tan^2(x)

Proof Complete

Prove   sin^3(x)cos^4(x)={cos^4(x)-cos^6(x)}sin(x)

Proof

sin^3(x)cos^4(x)=
sin^2(x)cos^4(x)sin(x)=
{1-cos^2(x)}cos^4(x)sin(x)=
{cos^4(x)-cos^6(x)}

Proof Complete

Prove each of the following

a) tan^5(x)=tan^3(x)sec^2(x)-tan^3(x);
b) sec^4(x)tan^2(x)={tan^2(x)+tan^4(x)}sec^2(x)
c) cos^3(x)sin^2(x)={sin^2(x)-sin^4(x)}cos(x)
d) sin^4(x)+cos^4(x)=1-2cos^2(x)+2cos^4(x)
e) Do Example 12 without looking at the provided solutions

Trigonometry Equation Solving

When a trigonometry equation is solved analytically, (using Algebra techniques), it can be sanity checked (partially verified) by making use of graphing. For example if you are assigned to find the solutions of the equation cos(3x)+sin(2x)=0 in the interval [0,2pi], once you have found these 0's, you could check your work by graphing the function y=cos(3x)+sin(2x) over the interval [0,2pi]. Where this graph crosses the x axis are the solutions to the equation cos(3x)+sin(2x)=0. This is not an absolute verification, since graphing does not give exact points where zero crossings occur.

Of course another way to check to see if you got the correct answers is to substitute the answers back into the original equation and see if the result is 0. Due to round off errors, a correct answer substituted into an equation may not be exactly 0, but it will be VERY close. This method is not a certain verification either.

Example 1

[Note: This example has more details of
[how to solve this type of problem than
[other examples in section. It is
[important that this example be studied.

Solve 2sin(x)-1=0

Solution

2sin(x)-1= 0 ->
2sin(x)= 1 ->
sin(x)= 1/2 ->
x=invsin(1/2)=pi/6 is one solution.

Over the domain of [0,2pi] i.e. from 0 to 2pi, graph y= sin(x) and then graph y= 1/2 on the same graph. Where the two graphs intersect, are the solutions of this equation. We note that x= pi/6 and x= 5pi/6 are two solutions within 1 period. Since sine is periodic (with a period of 2Pi) there are infinitely many solutions, and these solutions are pi/6 + n*pi and 5pi/6 + n*pi, where n = 0, +/- 1, +/- 2, +/- 3 ...

Solve the following equations
Make use of graphing to check answers.

a) 2cos(x)+1=0
b) sqrt(2)sin(x)+1=0
c) sqrt(3)sec(x)-2=0
d) cot(x)+1=0
e) Do Example 1 without looking at the provided solution

Example 2

Solve 3tan^2(x)=1

3tan^2(x)=1 ->
tan^2(x)= 1/3 ->
tan(x)= +/- sqrt(1/3) ->
x=invtan{sqrt(1/3)}=pi/6 and x=invtan{-sqrt(1/3)}=-pi/6 or 5pi/6

Graph 3tan(x). It is seen that 3tan(x) has a period of pi. In the interval [0,pi], the solutions to this equation are x=pi/6 and x=5pi/6. Therefore all solutions to this equation are x=pi/6 + n*pi and x=5pi/6 + n*pi for n = 0, +/- 1, +/- 2, +/- 3 ...

Solve the following equations
Substitute all answers into equation to check your answer.

a) tan(x)+sqrt(3)=0
b) 2sin(x)+1=0
c) cos(x){cos(x)-1}
d) sin^2(x)=3cos(x)
e) Do Example 2 without looking at the provided solution

Example 3

Solve cot(x)cos^2(x)=2cot(x)

cot(x)cos^2(x)=2cot(x) ->
cot(x)cos^2(x)-2cot(x)=0
cot(x){cos^2(x)-2}=0

setting each of these factors to 0, we have

cot(x)=0 and cos^2(x)-2=0 ->
cot(x)=0 and cos(x)= +/- sqrt(2)

cot(x)=0 has a solution at x=pi/2, and no solution exists for cos(x)= +/- sqrt(2) because +/- sqrt(2) are both out of the range of the cosine function.

Graphing cot(x) shows it has a period of pi. Given this the solutions for cot(x)=0 are x=pi/2 + pi*n where
n = 0, +/- 1, +/- 2, +/- 3 ...

Therefore the solutions for cot(x)cos^2(x)=2cot(x) are
x=pi/2 + pi*n where n = 0, +/- 1, +/- 2, +/- 3 ...

Solve the following equations (check answers)

a) 3tan^3(x)=tan(x)
b) 2cos^2(x)+cos(x)-1=0
c) 2sin^2(x)=2+cos(x)
d) cos(x)+sin(x)tan(x)=2
e) Do Example 3 without looking at the provided solution

Example 4

Find all solutions of 2sin^2(x)-sin(x)-1=0 on the interval [0,2Pi]

2sin^2(x)-sin(x)-1=0 ->
{2sin(x)+1}{sin(x)-1} ->

setting each factor to 0 we have

2sin(x)+1=0 and sin(x)-1=0 ->
sin(x)= -1/2 and sin(x)=1

solutions are 7pi/6, 11pi/6 and pi/2

Solve the following equations over the interval [0,2pi]

a) 2sin^2(x)+3sin(x)+1=0
b) 4sin^2(x)=2cos(x)+1
c) 2sec^2(x)+tan^2(x)-3=0
d) Do Example 4 without looking at the provided solution

Example 5

Solve 2sin^2(x)+3cos(x)-3=0

Solution

2sin^2(x)+3cos(x)-3=0 ->
2[1-cos^2(x)]+3cos(x)-3=0 ->
2cos^2(x)-3cos(x)+1=0
{2cos(x)-1}{cos(x)-1}=0

setting each factor to 0, solutions in the interval [0,2pi] are found to be x=0, x=pi/3, and x=5pi/3.

since cos(x) has a period of 2pi, all solutions to this equation are
x=2n*pi; pi/3 + 2n*Pi; 5pi/3 + 2n*pi; n = 0, +/- 1, +/- 2, +/- 3 ..

a) If you were unable to do some problems in Example 4, do them now.
b) Do Example 5 without looking at the provided solution

Example 6

Solve cos(x)+1=sin(x) in the interval [0,2pi]

cos(x)+1=sin(x) ->
{cos(x)+1}^2=sin^2(x) ->
{cos(x)+1}^2=1-cos^2(x) ->
cos^2(x)+2cos(x)+1=1-cos^2(x) ->
2cos^2(x)+2cos(x)=0 ->
2cos(x){cos(x)+1}=0

setting each factor to shows

2cos(x)=0 and cos(x)+1=0 ->
cos(x)=0 and cos(x)=-1   therefore

x=pi/2,3pi/2 and x=pi

Because the original equation was squared, some of these solutions may be extraneous. Therefore each of these solutions need to be checked one at a time to see if they are valid in the original equation cos(x)+1=sin(x). Doing this, we see the correct answers are. x=pi/2 and x=pi, The graph of y=cos(x)+1-sin(x) confirms these answers are correct because this graph has 2 x intercepts, one at x=pi/2 and one at pi.

Find all solutions to the following equations in the interval [0,2pi]

a) csc(x)+cot(x)=1
b) 4sin(x)=cos(x)-2
c) {1+sin(x})/cos(x) + cos(x)/{1+sin(x)}=4
d) Check solutions of problems 1,2,3 using graphing.
e) Do Example 6 without looking at the provided solution

Example 7

Solve 2cos(3t)-1=0

2cos(3t)-1=0 ->
2cos(3t)=1 ->
cos(3t)=1/2

In the interval [0,2pi], we know 3t=pi/3 and 3t=5pi/3 are the only solutions. So in general, solutions to this equation are
3t=pi/3 + 2n*pi and 3t=5pi/3 + 2n*pi, dividing through by 3 we get

t=pi/9 + 2n*pi/3 and t=5pi/9 + 2n*pi/3
where n=0, +/- 1, +/- 2, +/- 3, ...

Solve the equations below (check answers)

a) 3tan(x/2)+3=0
b) tan^2(3x)=3
c) tan(3x){tan(x)-1}=0
d) cos(x/2){2cos(x)+1}=0
e) Check solutions of problems 1,3 using graphing.
f) Do Example 7 without looking at the provided solution

Example 8

Solve sec^2(x)-2tan(x)=4

sec^2(x)-2tan(x)=4 ->
{1+tan^2(x)}-2tan(x)=4 ->
tan^2(x)-2tan(x)-3=0 ->
{tan(x)-3}{tan(x)+1}=0 ->

x=arctan(3) and x=arctan(-1)=-pi/4

since tan(x) has a period of pi we have,

x=arctan(3)+n*pi and x=-pi/4+n*pi

n= 0, +/- 1, +/- 2, +/- 3, ...

Solve the following equations (check answers)

a) sec^2(x)+tan(x)=3
b) 2cos^2(x)+cos(x)-1=0
c) sin^2(x)+cos(x)+1=0
d) Check solutions of problems 1,2,3 using graphing
e) Do Example 8 without looking at the provided solution

9) Use graphing to approximate solutions for the following.
a) 2cos(x)-sin(x)=0; b) x*tan(x)-1=0; c) Does cos(1/x)=0 have a greatest solution? If so find it.

10) Write the following trigonometric expressions as algebraic expressions. a) cos{invcos(x)-invsin(x)}; b) sin{2invtan(x)}
c) cos(invsin(x)-invtan(2x)}

11) Find all solutions of the following equations in the interval [0,2pi]. Utilize Graphing to sanity check your results.
a) sin(x+pi/3)+sin(x-pi/3)=1; b) cos(x+pi/4)-cos(x-pi/4)=1;
c) tan(x+pi)+2sin(x+pi)=0.

12) If possible find algebraic solutions. Utilize graphing to approximate all solutions. a) 2cos(x)+sin(2x)=0; Hint: Use the sine addition formula. b) sin(2x)-sin(x)=0; c) 4sin(x)cos(x)=1.

13) Make use of appropriate sinusoidal addition identities to find all zeros of these functions in the interval [0,2pi].
a) y=sin(5x)+sin(3x); b) y=cos(2x)-sin(6x);
c) cos(2x)/{sin(3x)-sin(x)}, make use of graphing to verify these answers.

14) Make use of appropriate sinusoidal addition identities to prove that {cos(x)-cos(3x)}/{sin(3x)-sin(x)}=tan(2x)

15) a) Make use of factoring and then the sin^2(x)={1-cos(2x)}/2 identity to prove that sin^2(x)= 1/8{3-4cos(2x)+cos(4x)}.
b) Prove cos^4(x)-sin^4(x)=cos(2x)

16) Prove sec(2o)=sec^2(o)/{2-sec^2(o)}

17) Prove 1+cos(10y)=cos^2(5y)

18) sin(o)+cos(o)cot(o)=csc(o)

19) cos(o)/{1-sin(o)}=sec(o)+tan(o)

20) {1+cos(o)}/sin{o} + sin/{1+cos(o)} = 2csc(o)

^^^^^^^^^^^^^^^^^^^^^^^^^^

15 General Problem Set 2 (Advanced)

^^^^^^^^^^^^^^^^^^^^^^^^^^

1) Harmonic Motion Problem Set

1-1) A mass suspended on a spring is moving in a vertical sinusoidal motion. The vertical position of the mass as a function of time is according to the equation z= 4.8cos(2.1t) feet. a) What is the vertical span of the mass's movement? b) What is the period?; c) frequency? d) the amplitude? e) At times(s=seconds) = 94.57s, 50.33s, -206.0s, -129.8s, what are the mass's vertical positions and direction of movement? (f) List times the mass will be at the positions <0ft,3ft,4ft,-2ft,-4.5 ft> and moving upward. g) List the times the mass will be at these positions and moving downward.

1-2) An ocean bouy bobs up and down every 3.4 seconds in a vertical sinusoidal motion. The vertical span of the ocean bouys movement is 1.4 meters. At time = 0 the bouy is at its highest position. a) What is the frequency? b) the period? c) the amplitude? d) Using a cosine function, write an equation for the bouys vertical position as a function of time. e) Using a sine function, write an equation for the bouys position as a function of time.

1-3) An ocean bouy bobs up and down every 4.2 seconds in a vertical sinusoidal motion. The vertical span of the ocean bouys movement is 3 meters. At time = 0, the ocean bouy is 80cm above the midpoint of its vertical span and is moving downward. Derive an equation for the bouys vertical position using a cosine function.

1-4) An ocean bouy bobs up and down every 5 seconds in a vertical sinusoidal motion. The vertical span of the ocean bouys movement is 8 feet. At time = 1.2 seconds, the bouy is 3.4 feet below the midpoint of its vertical span and moving upward, derive an equation for the bouys vertical position using a sine function.

2) A polarizing filter for a camera contains two parallel plates of polarizing glass, one fixed and the other able to rotate. If o is the angle of rotation from the position of maximum light transmission, then the intensity of light leaving the filter is cos^2(o) times the intensity entering the filter. Find all angles o, 0°<=o<=360° (in degrees-minutes) so that the intensity of light leaving the filter is 70% of that entering.

3) Assume a flat earth. An airplane flies 650 miles from city A to city B at a bearing (orientation) of 48 degrees, From city B to city C, the plane flies 810 miles at a bearing of 115 degrees. Find the distance and the bearing from A to C.

4) Assume a flat earth. An airplane flies to a destination 360 miles away. If at first the airplane flies 40 minutes at a constant speed of 300 mph, at a heading that is 3 degrees off course, through what angle should pilot turn the airplane to correct the error?

5-1) Keeping in mind a problem previously done, .. Wanting to determine the height of a flag pole, you position yourself 176 feet from the base of the flag pole. Using an optical instrument you then determine the angle (from the horizontal) from the optical instrument to the top of the flag pole is 21°. The optical instrument is 5 feet above the ground. How high is the flagpole? Devise a practical method, (using ability to measure both distance and angles) to determine the height of a mountain. The following problem 5-2 provides the solution to this problem. Try to do this yourself before looking at the solution below.

5-2) A tourist measures the angle of elevation to a mountain top as 'a'. The tourist then travels a distance of 'd' along level ground towards the mountain and measures the angle of elevation to the mountain top to be 'b'. a) Prove the height of the mountain top is h= d/{cot(a)-cot(b)}. b) How far was the tourist from the mountain top when each of these measurements were made? ... Note: We are neglecting the height of the tourist, is this justified?

5-3) The angle (from the horizontal) from a tourist to a mountain top is 'a', the tourist then travels a distance 'd' along a straight level course towards the mountain through a tunnel and out the other side, at which time the angle from the tourist to the mountain top is 'b'. Prove the height of the mountain is h= d/{cot(a)+cot(b)}.

6) Do not make use of any inverse trigonometry functions when doing this problem. The following rays have a common end point and exist in the 1st quadrant. Ray A has a slope of 3, ray B has a slope of 1, ray C has a slope of 5. What slope does ray D need to have so that the angle formed by rays A and B equals the angle formed by rays C and D?

7) The diagonal of a cube and the diagonal one of the cubes faces meet at a common point. Find the angle made by these two diagonals.

8) The diagonal of cube and a segment where two of the cubes faces meet share a common point. What is the angle between these two segments?

9) An equilateral triangle has vertexes a,b and c. d is a point on ac and is twice as close to a as to c. e is a point on bc, and is twice as close to c as to b. f is the point where the segments bd and ae intersect. Prove that angle bfe is 60 degrees.

10) Taipei the capital of Taiwan is located at a latitude of 25° 05min North (of the equator), and at a longitude of 121° 32' East (of Greenwich England). Salt Lake City the capital of Utah is located at a latitude of 40° 47' North and at a longitude of 111° 57' West. The circumference of the earth is 24,900 miles. a) Determine a set of Cartesian coordinates for Taipei and also for Salt Lake City. b) What is the measure of angle Taipei - center of the earth - Salt Lake City? c) What is the shortest possible distance an airplane can fly in going from Salt Lake City to Taipei?

11) An astronaut tall enough to have eyes 6 feet above the ground is exploring a spherical asteroid 5 miles in circumference and gets lost from his space ship. The spaceship's beacon is 21 feet above the ground. When the astronaut is first able to see the spaceship's beacon over the horizon of the asteroid, how far will he have to walk to get to the spaceship?

12.15) The distance from an observer to the closest point p0 on a sphere is 17 feet. p1 is an other point on the sphere 17 feet 4 inches away from the observer and angle p1-observer-p0 is 8 degrees. What is the radius of the sphere? Note: The path from the observer to the point P1 does not cross through any portion of the sphere.

13.15) a) Prove that the radius of the largest circle that will fit inside a triangle with sides of length A,B and C is r= (2*Area of Triangle)/(A+B+C); b) A triangle has sides of length A,B and C, calculate its area; c) Derive a formula for the radius of the largest circle that will fit inside this triangle in terms of A,B and C.

14) A manufacturer needs to place 9 identical ball bearings a inside a circular ring such that each ball bearing touches the circular ring and two other ball bearings. The inner diameter of the circular ring is 11cm. What diameter do the ball bearings need to be? Hint: See the previous problem.

15.15) An isosceles triangle has two sides of length 5. The angle opposite the third side is 40 degrees. Without making use of the area of this triangle, determine the radius of the largest circle that will fit inside this triangle.

16) The two non parallel sides of a trapezoid have the same length as the shorter parallel side. Prove that an angle of this trapezoid formed by the longer parallel side and a non parallel side, is bisected by a diagonal of the trapezoid. (Do not make use of inverse trig functions to solve this problem).

17.15) Prove: Every point of the circle, {2*cos(o),2*sin(o)} is twice as close to the point (1,0) as to the point (4,0).

18.15) a) Determine a parametric expression (see previous problem) of a circle such that each of its points are c times as close to the point (a,0) as to the origin. b) Verify that the points of this circle are c times closer to (a,0) than to the origin.

19) C is the center of a circle. A and B are points on this circle such that angle ACB <= 180° and ACB' >= 180°. (angle ACB + angle ACB' = 360°). D is a point that lies on this circle. If D is in the interior of angle ACB!, prove: angle ADB is 1/2 of angle ACB.

20-1) A ray is oriented at an angle of o, (0°<o<90°). The slope of this ray is 6. What is the slope of a ray with orientation of o/2?

20-2.15) Make use of the tangent addition formula to derive a tangent half angle formula which makes use of only tangent function(s) and not cosine functions.

20-3) Make use of the tangent half angle formula derived in the previous problem to do the following. An angle o (0°<o<90°) has a slope of 6. What slope does the angle o/2 have? Does your answer match the answer of problem 20-1?

20-4) Prove that the tangent half angle formula you derived in problem 20-2.15 {which makes use of tangent function(s) not cosine functions} is equal to the standard tangent half angle formula, (which makes use of cosine functions). [Let this proof be a verification of your derivation in problem 20-2.15].

21) Make use of Classical Geometry to prove the cosine subtraction formula, i.e. prove cos(a-b)=cos(a)cos(b)+sin(a)sin(b) where (0°<a<180°), (0°<b<90°) and (0°<a+b<90°).

22) Make use of Classical Geometry to prove the sine addition formula, i.e. prove sin(a+b)=sin(a)cos(b)+cos(a)sin(b) where (0°<a<90°), (0°<b<90°) and (0°<a+b<90°).

23) a) Find all cube roots of -1 + sqrt(3/2)i. b) Using the standard form of each of the numbers calculated in part 'a', verify your answers are correct.

24) a) A physics teacher desires to demonstrate the concept of beats to his students. He decides he would like to produce the note middle C (440Hz) such that it goes from loud to soft to loud again, once every second. Write up a set of requirements for a set of tuning forks which will fulfill these requirements. b) Prove all trigonometry identities you make use of to do this problem.

25.15) Determine an inverse function of a suitably domain restricted form of 3cos(x)+4sin(x).

26) a) What sinusoidal needs to be added to y=cos(t) so that the resultant function will be y=cos(t) shifted 13 degrees to the right? b) What sinusoidal needs to be added to y=sin(3t) so the resultant function will be y=sin(3t) with a decrease in amplitude of 20% and a delay of 1/12th of a second? (time is in seconds). c) Make use of graphing to sanity check your answers.

27.15) An arched doorway is in the shape of circular arc centered on top of a rectangle four feet wide and eight feet high. The curricular arc is 5 feet long. What is the area of the doorway?

28.15) Leaning Ladder Problem
a) Two buildings, A and B stand next to each other forming an alleyway between them. Two ladders, one 3m (meters) long and the other 4m long are in the alleyway. The bottom of the 3m ladder touches the base of building A and leans over against building B. The bottom of the 4m ladder touches the base of building B and leans over onto building A. The point where the ladders cross, is 1m above the ground. What is the width (w) of the alleyway?
b) Express the equation that defines the width of the alley in algebraic form.

29) If f(-x)=f(x), then f is an even function. If f(-x)=-f(x) then f is an odd function. Every Function (defined on the entire x axis) can be decomposed into an even function added to an odd function. Virtually any periodic function can be decomposed into a summation of sinusoidal functions, [see Fourier transform]. Virtually any non periodic function can also be decomposed into a summation of sinusoidal functions, [see Fourier integral]. In Engineering, it is often useful break down functions into their sinusoidal components so that systems which are driven by forces or electrical currents or voltages can be analyzed.

Even functions can be decomposed into a summation of cosine functions. Odd functions can be decomposed into a summation of sine functions. If a function is neither even or odd, the function can be decomposed into a summation of cosine functions and sine functions.

I) Every Function (defined on the entire x axis) can be decomposed into an even function + an odd function. a) Prove {f(x)+f(-x)}/2 is an even function; b) Prove that {f(x)-f(-x)}/2 is an odd function; c) Make use of what is proved in 'a' and 'b' to prove that any function defined on the entire x axis is the sum of an even function and an odd function.

II) A function y=f(x) has the value of: 1 between x=-2 and x=4, 3 between 4 to 5 and is zero every where else. Decompose this function into its even and odd components and then describe these even and odd functions in words and graph each of them.

Make use of the theorem (see Law of Cosines Law of Sines section)

The radius r of the circle passing through (circumscribing) all the vertices of any triangle is the length of any one of the sides of the triangle divided by, twice the sin of the angle opposite this side.

and the theorem

The area of a triangle is 1/2 of the product of the length of any two sides times the sine of the included angle.

to prove

30.15) Prove: The radius of a circle circumscribing a triangle with sides A,B,C is (A*B*C)/(4*Area of Triangle).

Preliminary to 31.15) Prove ..

A*cos(wt)+B*cos(wt)+C*sin(wt)+D*sin(wt)=E*cos(wt)+F*sin(wt) implies A+B=E and C+D=F

31.15) a) Where A<a, B<b, C<c are vectors in polar form, prove ..
A*cos(wt-a) + B*cos(wt-b) = C*cos(wt-c) implies A<a + B<b = C<c;
b) Prove the converse. c) Determine ? and ??,
3cos(wt-30°) + 4cos(wt+45°) = ?cos(wt-??).

32) In maps and navigation, angle orientations are defined differently than in mathematics. On a map, a positive angle of orientation or rotation is clockwise, not counter clockwise as is the case in mathematics. On a map; 0 degrees is straight up or North, 90 degrees is to the right or East, 180 degrees is downward or South. 270 degrees is to the left or West.

Neglecting curvature of the earth, solve the following problem. An observer in a light house is 347 feet above sea level and spots a ship in a direction (orientation) of 58 degrees 18 minutes and at an angle of depression from the horizontal of 1 degree, 12.6 minutes. The observer spots another ship at a direction of 341 degrees, 28 minutes and at an angle of depression from the horizontal of 2 degrees, 37.3 minutes. From the perspective of the first ship, what is the direction and distance of the second ship?

Does neglecting curvature of the earth make much of a difference?

(Do not make use of coordinates when doing this problem).

Rotating Magnetic Fields

Some magnets are shaped like a C where the ends of the C face each other. The ends of the C are the poles of the magnet. The magnetic flux of the magnet (by convention) exits the north pole of the magnet and enters the south pole. An electromagnet built in this manner has wire wound around the body of the magnet. The amount of magnetic flux exiting the north pole and entering the south pole is proportional to the current in the wire that is wound around the body of the magnet. If the voltage applied to the wire is sinusoidal, the intensity of the magnetic flux between the faces of magnetic the will also be sinusoidal.

Magnetic fields add like vectors. For example if at a point; magnet 1 creates a magnetic field in the positive x direction with a strength of 1, and magnet 2 creates magnetic field in the positive y direction with a strength of 1. Both magnets working together create a magnetic field at that point with a strength of sqrt(2) at an angle of 45 degrees.

33) If two electromagnets are situated so the magnetic flux each creates is at a right angle with respect to the flux the other creates, and the flux 'B', created by one of the electromagnets is $B1(t)=\cos(t)$, and the flux created by the other electromagnetic is $B2(t)=\sin(t)$ prove that both electromagnets working together produce a rotating magnetic field of constant strength.

Electricity provided by the power company is alternating current, or AC current. The magnitude of the current in each of the wires is a sinusoidal function with respect to time. The frequency of this current is 60 cycles per second. The power company provides 3 different phases of current. Certain industrial customers make use of all three phases of current. If the voltage of one of these phases is expressed as $V1(t)=A*\cos(t)$, the voltage of another phase could be expressed as $V2(t)=A*\cos(t-120°)$ and the voltage provided by the third phase could be expressed as $V3(t)=A*\cos(t-240°)$.

One use of 3 phase current is to power a certain type of electric motor that creates and then makes use of a rotating magnetic field. This electric motor has 3 (coplanar) electromagnets. If one of these electromagnets has an orientation of 0 degrees, the second would have an orientation of 120 degrees and the third would have an orientation of 240 degrees.

34) Three electromagnets are situated so the magnetic flux each creates makes a 120 degree angle with respect to the magnetic flux of either of the other magnets. The strength of the magnetic field as a function of time created by these magnets is $B1(t)=\cos(t)$, $B2(t)=\cos(t-120°)$, and $B3=\cos(t-240°)$. Prove such a setup can produce a rotating magnetic field of constant strength.

35.15) A and B are sides of a triangle, a and b are the corresponding (opposite) angles. Prove the Law of Tangents, i.e. prove

$$\frac{A-B}{A+B} = \frac{\tan\{(1/2)(a-b)\}}{\tan\{(1/2)(a+b)\}}$$

36) Make use of the Law of Tangents to solve the following SAS triangle problem for the angle(s) and side(s) not given.
$S1=3$, $A3=32°$, $S2=5$.

37) The following are Mollweide's equations. The capital letters are sides of a triangle, the small letters are the corresponding (opposite) angles. a) Prove these equations. b) Use either one of these equations to determine if 4,5,6,48°,61°,71° are sides and angles of a triangle.

$$\frac{A-B}{C} = \frac{\sin\left(\frac{a-b}{2}\right)}{\cos\left(\frac{c}{2}\right)} \quad : \quad \frac{A+B}{C} = \frac{\cos\left(\frac{a-b}{2}\right)}{\sin\left(\frac{c}{2}\right)}$$

38.15) Prove: If a,b,c are angles of a triangle, then
$\tan(a)+\tan(b)+\tan(c) = \tan(a)*\tan(b)*\tan(c)$

39.15) Napoleon's Theorem 1
If the sides of an arbitrary triangle are also sides of 3 different equilateral triangles and these equilateral triangles are pointed outward with respect to the arbitrary triangle. Prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. This equilateral triangle is referred to as the Napoleon triangle 1 of the arbitrary triangle.

40) Napoleon's Theorem 2
If the sides of an arbitrary triangle are also sides of 3 different equilateral triangles and these equilateral triangles are pointed inward with respect to the arbitrary triangle. Prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. This equilateral triangle is referred to as the Napoleon triangle 2 of the arbitrary triangle.

41) Napoleon’s Theorem - Area Difference
Given an arbitrary triangle, prove that the difference of the areas of its Napoleon triangle 1 and its Napoleon triangle 2 equals the area of the original arbitrary triangle. (This is a difficult problem to do using coordinate geometry techniques, much easier using trigonometry).

42) Make use of the unnamed theorem that is the basis of de Moivre's formula, i.e.

If A and B are complex numbers, then
magnitude of A*B = magnitude of A * magnitude of B
argument of A*B = argument of A + argument of B
i.e. A<a * B<b = (A*B)<(a+b) . . . {where < means angle}

to factor any of the following.

a) x^2+Y^2=x^2+Y^2

b) x^2-y^2=(x+y)(x-y)

-----

c) x^3+y^3=(x^2-xy+y^2)(x+y)

d) x^3-y^3=(x^2+xy+y^2)(x-y)

-----

e) x^4+y^4={x^2+sqrt(2)xy+y^2}{x^2-sqrt(2)xy+y^2}

f) x^4-Y^4=(x^2+y^2)(x+y)(x-y)

-----

g) x^5+y^5={x^2-2cos(Pi/5)xy+y^2}{x^2-2cos(3Pi/5)xy+y^2}(x+y)

h) x^5-y^5={x^2-2cos(2Pi/5)xy+y^2}{x^2-2cos(4Pi/5)xy+y^2}(x-y)

-----

i) x^6+y^6={x^2+sqrt(3)xy+y^2}{x^2-sqrt(3)xy+y^2}(x^2+y^2)

j) x^6-y^6=(x^2+xy+y^2)(x^2-xy+y^2)(x+y)(x-y)

-----

k) x^8+y^8=[x^2+sqrt{2-sqrt(2)}xy+y^2][x^2-sqrt{2-sqrt(2)}xy+y^2]
[x^2+sqrt{2+sqrt(2)}xy+y^2][x^2-sqrt{2+sqrt(2)}xy+y^2]

l) x^8-y^8={x^2+sqrt(2)xy+y^2}{x^2-sqrt(2)xy+y^2}
(x^2+y^2)(x+y)(x-y)

-----

m) Factor x^7+y^7

n) Factor x^7-y^7

^^^^^^^^^^^^^^^^^^^^^^^^^^

17 Dot Product / Cross Product / Navigation

^^^^^^^^^^^^^^^^^^^^^^^^^^

Vectors and parametric equations are prerequisites for this chapter. Dot product and cross product are useful tools for solving a variety of physics problems. The dot product and cross product will also be used in this book to help solve navigation problems.

Dot Product

Where |V| is the magnitude or length of any vector V, the dot product of two vectors A and B is defined as

dot product (A,B) = A.B = |A|*|B|*cos(o)
where o (0°<=o<=180°) is the angle formed by A and B

Notice the dot product of two vectors is not a vector, but a number, or scalar. .. If we think of the vectors A and B as sides of a triangle, and the third side of the triangle as side C, and the angled opposite side C as o, applying the Law of Cosines and we have

C^2 = A^2+B^2-2ABcos(o) ->

2A*B*cos(o) = A^2+B^2-C^2

If we place the vertex of the triangle where A and B meet at the origin, and if A=(x1,y1) and B=(x2,y2) we have

2 * A.B =

2 * A*B*cos(o) =

(A^2+B^2-C^2) =

(x1-0)^2+(y1-0)^2 + (x2-0)^2+(y2-0)^2 - { (x2-x1)^2+(y2-y1)^2 } =

x1^2+y1^2 + x2^2+y2^2 - (x2^2-2*x1*x2+x1^2 + y2^2-2*y2*y1+y1^2) =

x1^2+y1^2 + x2^2+y2^2 -x2^2+2*x1*x2-x1^2 -y2^2+2*y2*y1-y1^2 =

2x1*x2 + 2y1*y2

Therefore

where A = (x1,y1) and B = (x2,y2)

A.B = x1*x2 + y1*y2

Dot Product Problem Set

Keeping in mind that the definition of the dot product of two vectors A and B is

dot product (A,B) = |A|*|B|*cos(o)

do the following problems.

1) Where A=(x1,y1) and B=(x2,y2), prove:
A.B = x1*x2 + y1*y2

2) Where A=(x1,y1,z1) and B=(x2,y2,z2), prove:
A.B = x1*x2 + y1*y2 + z1*z2

3) What is the dot product of the following sets of vectors.
a) (1,7)(2,8); b) (8,-9)(2,1); c) (1,-2,11) (3,5,1)

4) What is the angle (less than or equal to 180°) formed by the following sets of vectors? a) (1,3)(2,7); b) (5,-1)(-2,7); c) (1,7,2)(-5,12,8)

5) Which of the following vectors are perpendicular?
a) (14,-15)(35,6); b) (8,7)(-3,11); c) (6,5,1)(3,1,-23)

6) Assuming the following sets of vectors are two sides of a parallelogram, make use of the dot product to calculate the area of the parallelogram, In doing this do not use any inverse trigonometry functions (on your calculator). a) (1,2)(-4,7); b) (1,2,3)(2,-3,7)

7) In the General Problem Set 2 (Advanced) problem set the student was asked to find the distance between Salt Lake City, Utah and Taipei Taiwan. Using the dot product the answer is easier to find than using the law of cosines. Using the cross product, the direction from Salt Lake City to Taipei can be determined.

Taipei the capital of Taiwan is located at a latitude of 25° 5' North (of the equator), and at a longitude of 121° 32' East (of Greenwich England). Salt Lake City the capital of Utah is located at a latitude of 40° 47' North and at a longitude of 111° 57' West. The circumference of the earth is 24,900 miles. a) Determine a set of Cartesian coordinates for Taipei and also for Salt Lake City. b) What is the measure of angle
Taipei - center of the earth - Salt Lake City?
c) What is the shortest possible distance an airplane can fly in going from Salt Lake City to Taipei?

8-) Find a vector that is perpendicular to each of the following two vectors, (1,2,3)(11,-7,5).

9) Find a vector that makes a 32° angle with the following vector
a)(1,-4); b)(1,-2,3)

Cross Product

The cross product of two vectors A and B is an other vector C, where C is perpendicular to both A and B. The magnitude of C is the magnitude of A times the magnitude of B times the sine of the angle defined o (0°<=o<=180°) by A and B. A x B obeys the right hand rule making A x B one vector, not two possible vectors.

Right Hand Rule (How to determine direction of AxB)

Orient your palm, so it is perpendicular to the plane defined by vectors A and B. Keep your fingers straight and aligned with your palm. Place the base of your right palm at the base of A and align your middle finger along vector A such that your palm faces o the angle (0°<=o<=180°) defined by A and B. Now if your thumb and your middle finger form a right angle, your thumb is pointing in the direction of AxB. If you were to follow these instructions using your left hand. Your thumb would be pointing the opposite direction. Try to solve the following problem yourself before looking at the solution below.

Exercise) Make use of the dot product to find a vector perpendicular to both V1=(a,b,c) and V2=(d,e,f)

Now we make use of the dot product to find a vector perpendicular to V1=(a,b,c) and V2=(d,e,f). If V3=(x,y,z) is perpendicular to both V1 and V2, this implies V3.V1=0 and V3.V2=0. Therefore

V3.V1=0 and V3.V2=0 ->

ax+by+cz=0
dx+ey+fz=0 ->

We have 2 equations and 3 unknowns, therefore we assign the value of 1 to x and then solve for the other variables. This is legitimate because the x coordinate of V3 can be anything. But once it is set the other coordinates of V3 are fixed.

a+by+cz=0
d+ey+fz=0 ->

f(a+by+cz)=0
-c(d+ey+fz)=0 ->

fa+fby+fcz=0
-cd-cey-cfz=0 ->

fa-cd + y(fb-ce)=0 ->

y(fb-ec) = cd-fa ->

y= (cd-fa)/(fb-ec)

-----

a+by+cz=0
d+ey+fz=0 ->

e(a+by+cz)
-b(d+ey+fz) ->

ea+eby+ecz
-bd-bey-bfz ->

ea-bd + z(ec-bf) = 0 ->

z(bf-ec)=ea-bd ->

z= (ea-bd)/(bf-ec)
We have found a vector perpendicular to both V1 and V2, it is

V3 = (x,y,z) = {1, (cd-fa)/(fb-ec), (ea-bd)/(bf-ec)}

If we multiply V3 by fb-ec, we simplify it and it still will be perpendicular to V1 and V2, therefore we do this and in the process assign a new value (magnitude) to V3.

V3 = (x,y,z) = {bf-ec, cd-fa, ea-bd}

Remembering that we desire that V3=V1xV2
We test V3 to see of it obeys the right hand rule. If V3=V1xV2 obeys the right hand rule in one instance it will obey it in all instances. Therefore we will test to see if it does. If not we will re-orient it (multiply it by -1) so that it does. Proving that if V3=V1xV2 obeys the right hand rule in one instance it will obey it in all instances is beyond the scope of this book.

V1=(a,b,c), V2=(d,e,f), V3=(x,y,z)

Assume V1 is a vector along the positive x axis, i.e. (1,0,0)
Assume V2 is a vector along the positive y axis, i.e. (0,1,0)

Making use of the definition of cross product to take the cross product of V1 and V2, we see that V1xV2 is a unit vector aligned with the positive z axis, i.e.

z=(0,0,1).

If V1=(1,0,0) and V2=(0,1,0) we have

a=1, b=0, c=0, d=0, e=1, f=0 therefore

V3= {bf-ec, cd-fa, ea-bd} = (0*0-1*0, 0*0-0*1, 1*1-0*0} = (0,0,1)

Therefore V3 as defined does obey the right hand rule, no adjustment of V3 is needed, i.e. we don't need to multiply it by -1.

One last thing needs to be known, what is the magnitude of V3? According to the definition of the cross product, |V3|=|V1|*|V2|*sin(o) where o (0°<=o<=180°) is the angle defined by V1 and V2.

It turns out that the magnitude of V3 as defined here is |V1|*|V2|*sin(o). The proof of this fact is beyond the scope of this book at this time. Most likely many students could prove this for themselves if they wish. However the proof of this is left to the student at this time.

Therefore where

V1=(a,b,c), V2=(d,e,f)
V1 x V2 = {bf-ec, cd-fa, ea-bd}

Admittedly remembering this equation is not easy. Fortunately there is a form of this expression which is very easy to remember. First some preliminaries.

Perhaps you have learned how to calculate determinates of a matrix in algebra. If not we will discuss it now. Our discussion will be a minimalist discussion of the subject. For a more in depth discussion consult a college algebra book. A matrix is defined as a rectangular array of numbers. What we will do here stretches this definition, because we will be dealing with a matrix whose elements are vectors and numbers. Then we will take a determinate of such a matrix which will give us the expression of V1xV2.

A determinate is an operation performed on a matrix. For example

```
The determinate of the matrix 1 2 is 1*4-3*2= 4-6=-2
                              3 4
```

```
In general the determinate of a 2 by 2 matrix a b is ad-cb.
                                              c d
```

One possible way to take the determinate of a 3 by 3 matrix follows.

```
   (a b c)
det(d e f) = a * det(e f) - b * (d f) + c * (d e)
   (g h i)          (h i)       (g i)       (g h)
```

Notice that the 1st 2x2 matrix is taken from the 3x3 matrix by excluding the row and column of a and taking what is left. Notice the 2nd 2x2 matrix is taken from the 3x3 matrix by excluding the row and column of b and taking what is left. Notice that the 3rd 2x2 matrix is taken from the 3x3 matrix by excluding the row and column of c and taking what is left.

Before moving on, calculate the determinates of

$$
\begin{matrix} -1 & 3 \\ 2 & 4 \end{matrix} \qquad
\begin{matrix} 2 & 7 \\ 1 & 5 \end{matrix} \qquad
\begin{matrix} a & b \\ c & d \end{matrix} \qquad
\begin{matrix} 1 & 2 & -1 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{matrix} \qquad
\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}
$$

As a check on your work the answers for the first 2x2 matrix and the first 3x3 matrix are -10 and 10 respectively.

Now we need to introduce the vectors i,j,k.
i=(1,0,0), j=(0,1,0), k=(0,0,1).
i is the unit vector aligned with the positive x axis.
j is the unit vector aligned with the positive y axis.
k is the unit vector aligned with the positive z axis.

Some example calculations are provided below.

3 * i = (3,0,0) : -2 * j = (0,-2,0) : 7 * z = (0,0,7)

2i-6j+5k = (2,-6,5)

Be able to do these example calculations for yourself before continuing.

With this we now give an alternative expression of A x B.

Where V1 = (a,b,c) and V2 = (d,e,f)

$$
V1 \times V2 = \det \begin{pmatrix} i & j & k \\ a & b & c \\ d & e & f \end{pmatrix}
$$

To ensure there is no mis-understanding this expression will be expanded below.

V1 x V2 =

$$
i * \det\begin{pmatrix} b & c \\ e & f \end{pmatrix} - j * \det\begin{pmatrix} a & c \\ d & f \end{pmatrix} + k * \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} =
$$

(bf-ec)i - (af-dc)j + (ae-db)k =

(bf-ec)i + (dc-af)j + (ae-db)k =

(bf-ec, dc-af, ae-db)

Therefore

Where V1 = (a,b,c) and V2 = (d,e,f)

V1 x V2 = (bf-ec, dc-af, ae-db)

<u>Cross Product Problem Set</u>

1) Give a full definition of the cross product of two vectors.

2) Answer the following questions about i,j and k.
a) i=?; b) j=?; c) k=?; d) 2i-3j=?; e) k-4i=?; f) 3i-2j+7k=?

3) Where V1=(a,b,c) and V2=(d,e,f) make use of

$$V1 \times V2 = \det\begin{pmatrix} i & j & k \\ a & b & c \\ d & e & f \end{pmatrix}$$

to show that

V1 x V2 = (bf-ec, dc-af, ae-db)

4) Prove A x B = -(B x A)

5) In the Dot Product problem set, the student was asked make use of the dot product to find the area of parallelograms. Now the student is asked to make use of the cross product to find the area of the same set of parallelograms. As a check on your work, compare your answers with the answers you got last time.

Assuming the following sets of vectors are two sides of a parallelogram, make use of the cross product to calculate the area of the parallelogram. a) (1,2)(-4,7); b) (1,2,3)(2,-3,7)

6) Make use of the cross product to find a vector perpendicular to the following planes. Remember two intersecting lines can be used to identify a plane. Likewise two non-collinear vectors identity a plane. Also 3 points define a plane.
a) V1 = (2,-8,1), V2 = (-5,3,4)
b) P1 = (1,-7,2), P2 = (6,-1,3), P3 = (-1,0,-2)
c) P1 = (a,b,c), P2 = (d,e,f), P3 = (g,h,i)

7) a) Make use of the dot product to find a vector V3 perpendicular to both V1 = (e,f,g) and V2 = (k,l,m). b) Make use of the vector you calculated in part 'a' to find another vector V3' perpendicular to V1 and V2 such that there are no denominators in the expression of V3'. c) Verify that V3' is perpendicular to both V1 and V2. c) Determine if V3' obeys the right hand rule or the left hand rule.

8) Where V1= (a,b,c), V2= (d,e,f), V1 x V2 = (bf-ec, dc-af, ae-db), o is the angle formed by V1 and V2, prove
|(bf-ec, dc-af, ae-db)| = |(a,b,c)| * |(d,e,f)| * sin(o)
note: |(a,b,c)| is the magnitude of vector (a,b,c).

Navigation Preliminaries

For most of the worlds history until the renaissance, ships had to stay within sight of land or they would get lost at sea. Hundreds of years ago it became possible to sail on the open seas, not within sight of a shoreline. The mathematics that helped to make this possible was (and is) spherical trigonometry. Spherical Trigonometry has had a major impact on the course of world history by helping to make worldwide navigation of ships and airplanes possible.

Spherical Trigonometry deals with triangle problems, distances, and orientations (or directions) on a sphere, which is a close approximation for the surface of the earth. This book will not cover spherical trigonometry. It turns out navigation problems can also be solved by making use of the dot product and the cross product and in this chapter teaches how to do exactly this. The two basic navigation problems are the following. I) If the longitude and latitude of two points on earth are known, what is the distance (going along the surface of the earth) between these two points?, and what is the direction from one point to the other? II) If a ship sails from A to B along the shortest possible course and another ship sails from C to D along the shortest possible course, what is the location where their paths cross?

Parametric equations is a prerequisite for this section. Here is a quick introductory review of parametric equations of lines. The line with slope m, passing through the origin i.e. y=mx can be expressed in parametric form as t(1,m) or (t,mt). The line y=mx+b can be expressed in parametric form as t(1,m)+(0,b). The line with slope m, passing through the point (a,b) i.e. y-b=m(x-a) can be expressed in parametric form as t(1,m)+(a,b). In three dimensions a line doesn't have a slope it has an orientation. A line in space passing through the origin and the point (a,b,c) has an orientation of (a,b,c) and a parametric form of t(a,b,c). A line passing through the point (x1,y1,z1) with an orientation of (a,b,c) is expressed in parametric form as t(a,b,c)+(x1,y1,z1) or as (ta+x1,tb+y1,tc+z1).

For every point in a plane or in space, there is a corresponding vector and visa versa. (a,b,c) is an expression for either a point or a vector, which it is depends upon context. The tail of any vector (a,b,c) is the origin or $(0,0,0)$. The head of the vector is the point itself or the point (a,b,c). Keeping this in mind we can more easily derive an equation of a plane in space.

The set of vectors that are perpendicular to a given vector in space lie on one plane. This set of vectors, or corresponding points make up a plane. Making use of the dot product to find the equation of the plane that is perpendicular to the vector $(1,2,3)$ we have $(1,2,3).(x,y,z)=0$ or $x+2y+3z=0$. The set of points (x',y',z') that satisfy equation $x+2y+3z=0$ make up the plane that is perpendicular to the vector $(1,2,3)$. ($ax+by+cz=0$ is the equation of the plane that is perpendicular to the vector (a,b,c).

Just as a line on a plane has a slope, a plane in space has an orientation. If a plane (P) in space is perpendicular to a vector V, then the orientation of P is defined to be the orientation of V. Two vectors have the same orientation (or direction) if the ratio between their elements is the same. For example, the orientation of (a,b,c) = orientation of (ka,kb,kc) because $a/b = (ka)/(kb)$, $b/c = (kb)/(kc)$, $a/c=(ka)/(kc)$ ... etc. (where k is any real number not equal to 0). If vector V equals (a,b,c) it is rightfully said the orientation of V is (a,b,c).

A line with a slope of m containing the origin can be represented by the equation $y=mx$. The line with a slope of m that passes through the point $(x1,y1)$ has an equation of $y-y1=m(x-x1)$. Likewise the plane containing the origin with an orientation of (a,b,c) has the the equation $ax+by+cz=0$ and a plane with an orientation of (a,b,c) that passes through the point $(x1,y1,z1)$ has an equation of $(x-x1)+b(y-y1)+x(z-z1)=0$. The proof of this is left to the student. Hint: See the proof of .. The function $y=f(x)$ shifted 'a' to the right is $y=f(x-a)$ in Chapter 10 Graphing Trigonometry Functions and their Inverses.

If two intersecting planes in space are viewed from an 'edge on' perspective, they appear as two intersecting lines forming two sets of opposite angles. If one set of opposite angles has measure o, the other has measure 180°-o, and the two planes form two sets of opposite dihedral angles, one having measure o, the other having measure 180°-o. To determine the angle o (0°<o<=90°) formed by the planes, first determine their equations. Use the equations to determine a vector perpendicular to each plane. Then make use of either the dot product or the cross product to determine the angle o (0°<o<=90°) formed by these vectors. o (0°<o<=90°) is also the angle formed by the planes, the proof of this is left to the student. In doing this consider the intersecting planes and the vectors perpendicular to these planes from an 'edge on' perspective.

Navigation Preliminaries Problem Set

1) a) Give a Cartesian equation and a parametric equation of a) a line with slope of m, passing through the origin. b) a line with slope of m passing through the point (a,b).

2) What is the orientation of a line passing through a) the origin and the point (a,b,c)? b) the points (a,b,c)(d,e,f)

3) What is the parametric equation of the line a) passing through the origin with an orientation of (a,b,c); b) passing through the point (q,r,s) with an orientation of (n,v,j)?

4) What is the orientation and parametric equation of the line passing through the points (a,b,c)(d,e,f).

5) Make use of the dot product to derive the equation of a plane perpendicular to the vector (a,b,c).

6) What is the equation of a plane that is perpendicular to the vector (a,b,c) and that passes through the point (d,e,f).

7) Give an equation of a plane that contains the line a) t(a,b,c); b) t(a,b,c)+(d,e,f) {expressed another way as) (at+d,bt+e,ct+f)}.

8) Give an equation of a line contained by the plane a) ax+by+cz=0; b) ax+by+cz=1

9) How many planes passing through a given point on a line are perpendicular to the line?

10) How many planes passing through a point located on a plane are perpendicular to the plane?

11) Give 'the' equation of the plane that passes through the following point and is perpendicular to the following vector. Put the equations in standard form, for example -x+12y+3z=10 is in standard form.
a) Point = (0,0,0) : vector=(3,-2,5)
b) Point = (1,-2,3) : vector=(4,-2,5)
c) Point = (d,e,f) : vector=(a,b,c)

12) Give 'the' equation of the plane that contains the following two vectors. a)(-7,8,1)(1,3,7); b)(a,b,c)(d,e,f)

13) Give the equation of the plane that passes through the following three points. a)(1,2,3)(7,6,4)(1,1,1); b)(a,b,c)(d,e,f)(h,i,j)

14) Give the parametric equation of the line that is the intersection of the following two planes. -2x+4y-7z=0 : x+y+2z=0

15) Give the parametric equation of the line that is the intersection of the following two planes. x+y+z=9 : 6x-2y-x=4

16) Give the parametric equation of the line that is the intersection of the following two planes. (Three points 'define' a plane). (-11,2,4)(2,3,4)(1,2,3) : (-2,1,7)(9,8,3)(5,-12,-1)

17) Prove: The angle 'a' (0° < a < 90°) formed by two intersecting planes P1 and P2 equals the angle b (0° < b < 90°) formed by two vectors V1 and V2, where V1 is perpendicular P1 and V2 is perpendicular to P2.

18) Intersecting planes form 2 sets of opposite (dihedral) angles. The difference of orientation of intersecting planes is the measure of 'the' angle (less than or equal to 90 degrees) formed by them. Parallel planes have a difference of orientation of 0. What is the difference of orientation of the following sets of planes. a) -7x+8y+9z=0 and 3x+2y+z=0; b) 7x+y+2z=4 : -x+2y+9z=7 c) (-11,2,4)(2,3,4)(1,2,3) : (-2,1,7)(9,8,3)(5,-12,-1)

19) Find a plane whose orientation is 32° different than the plane 7x+8y+z=0.

20) Prove: The difference of orientation of ax+by+cz=0 : dx+ey+fz=0 and ax+by+cz=m : dx+ey+fz=n is the same.

21) In this problem to make the visualization easier, assume the planes Pa and Pb pass through the origin. If vector Va is perpendicular to plane Pa and if vector Vb is perpendicular to plane Pb a) Prove Va and Vb are perpendicular to the line that is the intersection of Pa and Pb. b) Prove that Va x Vb is collinear to the line of intersection of Pa and Pb. (Hint: This proof is probably easier to do using geometrical techniques, rather than analytic techniques. Your choice).

Navigation

The intersection of a plane and a sphere is a circle. If a plane passes through the center of a sphere, the intersection of the plane and sphere is referred to as a great circle. All great circles of a sphere and the sphere itself have the same center. Given any two points A and B of a sphere, there is a great circle of the sphere that passes through these two points. This is because there is a plane that contains these two points and the center of the sphere, and the intersection of this plane and the sphere is a great circle that contains (passes through) A and B. The shortest path between any two points of a sphere is a path along the great circle that connects the two points. Any two great circles of a sphere, bisect each other. This is because .. the planes that contain these great circles intersect in a line that passes through the point that is the center of both great circles (and the sphere) (why?), therefore the two points where this line intersects the sphere are endpoints of a diameter of the sphere and of both great circles (why?). The endpoints of a diameter of any circle, bisect the circle (why?). Therefore two great circles of a sphere bisect each other (why?). Any great circle which intersects one pole of the earth (north pole or south pole) intersects both poles. (Prove or justify this for yourself). Such a great circle is called a polar great circle.

In maps and navigation, angle orientations are defined differently than in mathematics. On a map; 0 degrees is straight up or North, 90 degrees is to the right or East, 180 degrees is downward or South. 270 degrees is to the left or West. Also on a map, a positive angle of orientation or rotation is clockwise, not counter clockwise as is the case in mathematics. At a given point on the earth, a vector pointed north and a vector pointed east form an angle of 90 degrees. Two vectors at different places on the earth, pointed in a direction of (for example) of 17 degrees, do not necessarily have the same orientation. Take a few moments to see why this is true.

What does it mean if we say the direction from point A to point B is (for example) 17 degrees? Does it mean if you are at point A, that in order to get to point B, you move constantly in the direction of 17 degrees until you arrive at point B? No it doesn't. Remember the shortest distance between any two points on the surface of a sphere is along the path of a great circle and in general when traveling along a great circle of a sphere you are constantly changing directions. The two exceptions to this are polar great circles and the equator. If the direction from A to B is for example 17 degrees, what this means is if you are traveling along the shortest path from A towards point B, you begin your journey by moving in the direction of 17 degrees, and from there you will constantly change your direction in order to stay on the great circle which is the path that is the shortest path between A and B.

In navigation problems it is useful to be able to determine the orientation of a plane with respect to another plane (from a given perspective). To visualize what this means, view two intersecting planes, A and B, from an 'edge on' perspective. In doing this A and B appear as two lines intersecting at a point. By definition (from this perspective) B has an orientation of o ($-90° < o <= 90°$) with respect to A if A rotated clockwise by an angle of o (about the line of intersection of A and B) causes A to have the same orientation as B. For example if rotating A .. 5 degrees clockwise would causes A to have the same orientation as B, then B has an orientation of 5 degrees with respect to A (from this perspective). Another example, if rotating A -5 degrees clockwise would cause A to have the same orientation as B, then B has an orientation of -5 degrees with respect to A (from this perspective). (Remember a rotation of -5 degrees clockwise is a rotation of 5 degrees counter clockwise).

In navigation problems it is also useful to be able to determine the orientation of a vector with respect to another vector (from a given perspective). This definition is similar to the definition of 'orientation of a plane with respect to another plane'. Given vectors Vo and V1, V1 has an orientation of o with respect to Vo if Vo rotated by an angle of o clockwise (about the origin) causes it to have the same orientation as V1.

In the sentence "In navigation problems it is useful to be able to determine the orientation of a plane with respect to another plane from a given perspective", why is it necessary to say 'from a given perspective'? The reason for this is that there are two possible 'edge on' views (perspectives) of two planes intersecting. For example if two planes intersect in a vertical line, one 'edge on' view of the two intersecting planes would be looking up, the other 'edge on' view of the intersecting planes would be looking down. If from one 'edge on' view (perspective) B has an orientation of +5 degrees with respect to A, then from the other perspective, B has an orientation with respect to A of -5 degrees.

A plane P1 from a given 'edge on' perspective has an orientation of o (-90° < o <= 90°) with respect to Po. If (vector) Vo is perpendicular to Po, and (vector) V1 is perpendicular to P1, then either V1 or -V1 has an orientation of o with respect to Vo. Verify this for yourself before reading further.

Let Po be the plane containing point A (located on the surface of the earth), the north pole, and the center of the earth. The intersection of Po and the sphere of the earth is a polar great circle, GCp. Anyone traveling along this great circle is always traveling north or south. Let P1 be the plane containing point A (on the surface of the earth), point B (on the surface of the earth) and the center of the earth. The intersection of P1 and the sphere of the earth is the great circle GC1 of the earth containing points A and B. Assuming the orientation of P1 with respect to Po is o, anyone traveling along great circle GC1, while at the point A, is traveling in a direction o or 180°-o where one of these directions is the direction from A to B. Before reading further, take a few moments to assure yourself these things are true.

To determine the orientation of plane P1 with respect to plane Po do the following. Consider the center of the earth as the origin. For simplicity sake consider the radius of the earth as 1. Consider the coordinates of the north pole as (0,0,1). You are given the longitude and latitude of points A and B (A and B are points on the surface of the earth). Make use of all this information to determine a set of 3 dimensional Cartesian coordinates of A and B. Remembering that Po contains the points, A, the north pole and the center of the earth, make use of all this information to determine the equation of Po. Make use of this equation to determine a vector Vo such that Vo is perpendicular to Po. Remembering that P1 contains the points A, B and the center of the earth, determine the equation of P1. Make use of this equation to determine a vector V1 such that V1 is perpendicular to P1. V1.Vo < -1 implies the angle o (o < 180°) formed by V1 and Vo is greater than 90°. If this is the case, multiply V1 by -1 and call this vector V1. Now we have V1.Vo > 1 and the angle o (o < 180°) formed by V1 and Vo is less than or equal to 90°. At this point if P1 has an orientation of o with respect to Po, then V1 has an orientation of o with respect to Vo. Assure yourself this is true before reading further. Make use of the dot product to determine |o|. Determine the sign of o as follows. If V1 x Vo is a vector pointing in the direction of A, then o is positive. (make use of the right hand rule to verify this for yourself). Mathematically this means k(V1 x Vo)=A for some k, where k is a positive number. If V1 x Vo is a vector pointing in the direction away from A, then o is negative. (Make use of the right hand rule to verify this for yourself). Mathematically this means k(V1 x Vo)=A for some k, where k is a negative number. Make use of this information to determine the sign of o.

If you have done these calculations correctly you have the angle o, where the direction from A to B is either o or 180°-o.

Following is an example of how one would determine if the direction from A to B is o or 180°-o.

Suppose the longitude / latitude of A is 2° west, 18° north.
Suppose the longitude / latitude of B is 51° west, 72° north.

The longitudinal difference from A to B going west is 49°
The longitudinal difference from A to B going east is 360°-49°= 311°

Going west is closer longitudinally than going east. Therefore the direction from A to B is a westerly direction. If o is a westerly direction then the direction from A to B is o. If 180°-o is a westerly direction then 180°-o is the direction from a to B. If the direction from A to B were strictly north or south, the method used would be similar.

A (heuristic) proof of why the shorter path between two points along a great circle is the path that transverses the least longitudinal distance follows. Consider the non polar but otherwise general great circle GC1 passing through two arbitrary points A and B on a circle. Consider the polar great circle GCp passing through A. Half of GCp has the same longitude as the point A, the other half of GCp has a longitudinal distance of 180° from A. The dividing points between these two halves are the north and south poles. It is accepted here without proof that traveling one way along GC1 is (always) traveling in a westerly direction, and that traveling the other way along GC1 is (always) traveling in an easterly direction. Great circles GC1 and GCp bisect each other. Therefore if one starts at A and travels half way around GC1, they will again be at a place where GC1 and GCp meet. At this point, GCp has a longitude that is 180° different from A, therefore this point of GC1 also has a longitude that is 180° different from A. Therefore: if one travels half way around a (non polar) great circle they will have traveled a longitudinal distance of 180°, and if one travels less than half way around a (non polar) great circle they will have traveled less than a longitudinal distance of 180°, and if one travels more than half way around a (non polar) great circle they will have traveled more than a longitudinal distance of 180°. Therefore if two points have a longitudinal separation of other than 180°, the path along the great circle which connects them, which transverses less than 180° longitudinally is shorter than the path along the great which connects them that transverses more than 180° longitudinally. Remembering that the shortest path, between two points on a great circle is a path along the great circle that connects them, the path along a great circle connecting two points that transverses the least longitudinal distance is the shortest path between the two points. Proof Complete

In old times, when navigating a sailing ship, once each day a sextant, a calendar and a clock were used to determine the position (longitude and latitude) of the ship. From this information and knowing the location of the next port of call, the direction the ship needed to sail the rest of that day was computed by an officer of the ship who knew spherical trigonometry. Sailing ships moved slow enough that determining the direction of travel once a day was generally sufficient. Today commercial airplanes fly much faster. Navigation equipment on the airplane allows the airplane to know its 'exact' location at all times. A computer on the airplane constantly computes (several times a minute), which direction the plane needs to fly to stay on course.

Navigation Problem Set

1) The following problem was given previously. Now in addition you are asked to find the direction from Salt Lake to Taipei.

   Taipei the capital of Taiwan is located at a latitude of 25° 5' North (of the equator), and at a longitude of 121° 32' East (of Greenwich England). Salt Lake City the capital of Utah is located at a latitude of 40° 47' North and at a longitude of 111° 57' West. The circumference of the earth is 24,900 miles. a) Determine a set of Cartesian coordinates for Taipei and also for Salt Lake City. b) What is the measure of angle Taipei - center of the earth - Salt Lake City? c) What is the shortest possible distance an airplane can fly in going from Salt Lake City to Taipei?

```
                          longitude / latitude
Miami Florida   . . . . .  82° 12' W / 25° 46' N
Freetown Serria Leone . .  13° 14' W /  8° 29' N
Caracas Venezuela . . . . .  66° 55' W / 10° 30' N
Viana do Castelo Portugal   8° 50' W / 41° 42' N
```

2) a) What is the distance from Miami Florida to Freetown Sierra Leone? b) What is the direction from Miami Florida to Sierra Leone? c) Write a calculator or computer program which will solve this type of a problem.

3) a) A ship is to sail from Miami Florida to Freetown Sierra Leone. On its way, it is scheduled to meet up with and exchange some cargo with another ship that is sailing from Viana do Castelo Portugal to Caracas Venezuela. Both ships intend to take (and not deviate from) the shortest route possible to their destination. What is the location of their meeting place? b) Write a calculator or computer program which will solve this type of a problem.

www.mecmath.net/trig/trigbook.pdf

pg 84
pg 86 10) 12)
Pg 49
---
pg 66 cos{arccos(a)+arccos(b)} = ?
cos{arccos(a)+arcsin(b)} = ?
cos{arcsin(a)+arcsin(b)} = ?
sin{arccos(a)+arcsin(b)} = ?
tan{arccos(a)+arcsin(b)} = ?

arrow of time discussion

----

pg 67 # 7 do solution

If you have questions, comments or suggestions for improvement of this book, please contact me. Put Trigonometry Book in the subject line.

Curtis Blanco
Bountiful UT

curtis_blanco@hotmail.com

^^^^^^^^^^^^^

Index

^^^^^^^^^^^^^

A*

AM radio .. 92

amplitude .. 73

arccos(x), arcsin(x), arctan(x)   (see invcos, invsin, .. etc.)
arcsec(x), arccsc(x), arccot(x) .. 64, 65

angles .. 10

B*

bandwidth .. 91

beats .. 87, 144 problem 24

C*

carrier wave .. 90

compression theorem, f(x) compressed by a factor of k -> f(kx) .. 74
compression then shifting theorem .. 77

complex number .. see number

cosecant // csc(x) .. 14

cotangent // cot(x) .. 14

cosine // cos(o)
- triangle definition .. 13
- circle definition .. 13

cosine is an even function cos(-x)=cos(x)
- classical geometry proof .. 35 problem 2 (student solution)
- analytic proof .. 39

cosine addition formula cos(a+b)=cos(a)cos(b)-sin(a)sin(b)
- classical geometry proof .. 35 problem 4.6.2
- analytic proof .. 40

cosine subtraction formula cos(a-b)=cos(a)cos(b)+cos(a)cos(b)
- classical geometry proof .. 144 problem 21 (student solution)
- coordinate geometry proof .. 37

cross product .. 152 (see dot product)

D*

degree .. 10, see note on page 18

deMoivre's Formula .. 118
theorem preceding deMoivre's Formula .. 117

direction (on a sphere)(on the earth) .. 161

domain .. 62 (see range)

dot product .. 150 (see cross product)

The doppler effect .. 61

E*

even function, f(-x)=f(x) .. 39, 145 problem 29 (see odd function)

F*

fourier integral .. 145 problem 29, 90, 92
fourier transform .. 145 problem 29, 90

frequency .. 59

G*

grads .. 10

great circle .. 160

H*

Harmonic Motion .. 141 problems 1

Heron's Formula .. 25 (given sides of a triangle, find its area)

I*

imaginary number .. see number

invcos(x), invsin(x), invtan(x)
invsec(x), invcsc(x), invcot(x) .. 64, 65

inverse function .. 62

J*

K*

L*

Law of Cosines .. 24
Law of Sines .. 26

longitude .. 164 problem 1
latitude .. 164 problem 1

lower side band (LSB) .. see sideband

M*

minute .. 10, see note on page .. 18

A mistake to avoid when doing proofs .. 50

modulation .. 85, 90

Mollweide's equation .. 148, problem 35

N*

number ..

complex number .. 111
- standard form .. 112, 117
- Cartesian form .. 112, 117
- polar form .. 115, 117
- magnitude of .. 115
- argument of .. 115
- conjugate of .. 112
- foundational properties of .. 113, 114 (in blue)

imaginary number .. 111

number plane .. 112

Napoleon's Theorem .. 148, problem 37.15

O*

odd function, f(-x)=-f(x) .. 39, 145 Problem 29 (see even function)

one dimensional graph .. 91

orientation .. 11

P*

period .. 59
periodic function .. 58
Pi .. 10
Polar coordinates .. 103
Polar functions .. 107

Pythagorean identity .. 16
- triangle proof .. 16
- circle proof .. 17

Q*
R*

range .. 62 (see domain)

radians .. 10
revolutions .. 10
right angle .. 10

radio theory .. 90

reflection of a point .. 71
reflection of a function .. 71

relative maximum of a function .. 79
relative minimum of a function .. 79

right hand rule .. 152 (see cross product)

Rotating Magnetic Field .. 147 problems 33 and 34

S*

secant // sec(o) .. 14

second .. 10, see note on page 18

sideband (radio transmission and reception)
lower sideband (LSB) .. 91
upper sideband (USB) .. 92
single sideband (SSB) .. 92

sine // sin(x)
-triangle definition .. 13
-circle definition .. 13

sine is an odd function sin(-x)=-sin(x)
- classical geometry proof .. 35 problem 3 (student solution)
- analytic proof .. 40

sine addition formula sin(a+b)=sin(a)cos(b)+sin(b)cos(a)
- classical geometry proof .. 144 problem 22 (student solution)
- analytic proof .. 41
sine subtraction formula sin(a-b)=sin(a)cos(b)-sin(b)cos(a)
- classical geometry proof .. 35 problem 5.6.2
- analytic proof .. 42

single sideband (SSB) .. (see sideband)

Size of the Earth .. 20 & 21, problems, 22,23,24

sinusoidal .. 73, 78

shifting theorem, f(x) shifted to the right by a becomes f(x-a)   76

sound .. 60

sound decomposition .. 90

spectrum .. 91

T*

tangent // tan(o)
-triangle definition .. 13
-circle definition .. 13

triangle solutions .. 20 problem 19.4, 22, 26-29

tuning musical instruments .. 88   (see beats)

law of tangents .. 148, problem 33.15

U*

upper sideband (USB) .. (see sideband)

V*

W*

wavelength .. 58

X*

Y*

Z*

Answers to Problems

IV.2) Use Classical Geometry to prove the 30°-60°-90° triangle theorem.

Draw an equilateral triangle ABC. Label the top vertex A. Label the bottom left vertex B. Label the bottom right vertex C. Introduce a segment from C to the midpoint of BC. Label this midpoint D. Since triangle ABC is an equilateral triangle, each of its angles are 60 degrees. Triangles ABD and ACD are congruent by SSS (Side, Side Side). Since angles BAD and CAD are corresponding angles of congruent triangles, they are congruent. Added together make an angle of 60 degrees, therefore <BAD=30° and <CAD=30°. Since triangles ADB and ADC are corresponding angles of congruent triangles they are congruent, since they are supplementary, <ADB=90° and <ADC=90°. Sides BD and CD are corresponding sides of congruent triangles, therefore these sides are congruent. If each side of triangle ABC has a length of 1. BA=1, CA=1, BD=1/2, CD=1/2, and by the Pythagorean theorem, AD = sqrt(3)/2. Therefore in a 30°, 60°, 90° triangle. If the side opposite the 90 degree angle has a length of 1, the side opposite the 30 degree angle has a length of 1/2, and the side opposite the 60 degree angle has a length of 1/2.

Conversely, if a triangle has angles of 30°, 60° and 90°. Its sides are proportional to any other triangle that has angles 30°, 60° and 90°. Therefore its sides are (1/2)*a, {sqrt(3)/2}*a, 1*a for a<0. Therefore if a=1, or if its longest side is 1, such a triangle would have sides of 1/2, sqrt(3)/2, 1.

1.4) A degree is 1/360th of a revolution. A minute is 1/60th of a degree. A second is 1/60th of a minute. a) Express the angle 17° 42' 18'' as an angle of degrees. (Retain at least 5 significant digits after the decimal point). b) Express the angle 76.43120 degrees, in degrees, minutes, seconds format.

There are several ways this could be done. The easiest methods known to me will be shown here.

To convert the angle 17° 42' 18'' seconds to degrees, start by converting the 18 seconds to minutes, i.e. divide 18 by 60. The number displayed by your calculator should be 0.3 (minutes. To this add 42 (seconds). The number displayed by your calculator should now be 42.3 seconds. To convert this number to degrees, divide it by 60. The number displayed by your calculator should now be 0.705. To this number add 17. The number displayed by your calculator should now be 17.705 (degrees). Therefore we have 17° 42' 18'' = 17.70500 degrees.

Convert the angle 76.43120 to degrees minutes seconds format as follows. 76 is the whole number of 76.43120. Therefore

degrees = 76

We have 76.43120 - 76 = 0.43120 degrees left.

To convert 0.43120 degrees to minutes multiply it by 60.
0.43120 degrees * 60 = 25.872 minutes, Therefore

minutes = 25

Now we have 25.872 - 25 = 0.872 minutes left.

To convert 0.872 minutes to seconds, multiply it by 60.
0.872 minutes * 60 = 52.32 seconds. Therefore

seconds = 52.32 or

seconds = 52

depending on the format you choose to use, therefore

76.43120 degrees = 76 degrees 25 minutes 52 seconds

With practice, you should get to the point where you can easily do this with your calculator, without writing anything on paper. I highly recommend getting a RPN (Reverse Polish Notation)calculator to do calculations. Once you learn to use an RPN, calculations are much easier, so is recovering from errors when making calculations. RPN calculators can be purchased from Hewlett Packard, or can be download to a smart phone.

2.2) A triangle has a hypotenuse of length 5. One of its angles is 23°. What are the lengths of the legs of this triangle?

The Picture

The hypotenuse we label C. The side opposite 23° we label A, the other side we label B.

First we find the length of the leg opposite the angle of 23°

$$\sin(23°) = \frac{A}{C} \quad \rightarrow \quad A = C * \sin(23°) = 5 * \sin(23°) = 1.9536..$$

Next we find the length of the leg opposite the other acute angle.

$$\cos(23°) = \frac{B}{C} \quad \rightarrow \quad B = C * \cos(23°) = 5 * \cos(23°) = 4.6025..$$

10.4) An arrow is shot at an angle of 37° from the horizontal, at a speed of 175 ft/sec. At the instant it is shot, a) What its horizontal speed? b) What is its vertical speed?

To determine the speed the instant it is shot, lets assume there is no gravity and no air resistance, so that the speed and direction of the arrow remains constant through out its flight. Taking away the effects of gravity and air will not change the initial speed or direction of the arrow.

Let 'a' be the place where the arrow is shot from. Let 'b' be the place where the arrow is one second after it is shot, Let 'c' be the location where a vertical line through the arrow at 'b' would intersects the ground. abc defines a right triangle. The distance the arrow travels during the first second equals the length of side ab or 175 feet.

$$\frac{ac}{ab} = \cos(37°) \rightarrow ac = ab * \cos(37°) \rightarrow$$

$$ac = 175 \text{ ft} * \cos(37°)$$

ac is the horizontal distance the arrow travels during its first second, implying that the average horizontal speed of the arrow during its first second is 175 * cos(37°) ft/sec. Since the speed and direction of the arrow are constant, this speed is also the initial horizontal speed of the arrow when it is shot. Therefore the initial horizontal speed of the arrow is

a) 175 * cos(37°) ft/sec

$$\frac{bc}{ab} = \sin(37°) \rightarrow bc = ab * \sin(37°) \rightarrow$$

$$bc = 175 \text{ ft} * \sin(37°)$$

bc is the vertical distance the arrow travels during its first second, implying that the average vertical speed of the arrow during its first second is 175 * cos(37°) ft/sec. Since the speed and direction of the arrow are constant, this speed is also the initial vertical speed of the arrow when it is shot. Therefore the initial vertical speed of the arrow is

b) 175 * sin(37°) ft/sec

11.4) Hint: See solution of problem 2.2.

19.4) Two ranger stations a and b are 38km apart and a straight road connects them. A fire f starts in the forest. Station a takes a measurement and notes that angle fab = 39°. Station b takes a measurement and notes that Angle fba = 74°. a) How far is it from each of the ranger stations to the fire? b) What is the distance from the road to the fire?

The Picture (the following paragraph and the problem = the picture).

Let c be the point on the road between a and b such that fc is perpendicular to ab. D is the length of segment fc. ? is the length of ac. ?? is the length of segment cb. A is the length of segment fb and B is the length of segment af.

$d = d \rightarrow$

$B * \sin(39°) = A * \sin(74°) \rightarrow$

$$B = A * \frac{\sin(74°)}{\sin(39°)} \quad \leftarrow \text{eq 1}$$

$? + ?? = 38 \rightarrow$

$$B * \cos(39°) + A * \cos(74°) = 38 \quad \leftarrow \text{eq 2}$$

Substituting equation 1 into equation 2 is

$$A * \frac{\sin(74°)}{\sin(39°)} \cos(39°) + A * \cos(74°) = 38 \rightarrow$$

$$A \left\{\frac{\sin(74°)}{\sin(39°)} \cos(39°) + \cos(74°)\right\} = 38 \rightarrow$$

Answers to part a) in the next two equations

$$A = \frac{38}{\left\{\frac{\sin(74°)}{\sin(39°)} \cos(39°) + \cos(74°)\right\}} = \underline{25.979..}$$

$$B = A * \frac{\sin(74°)}{\sin(39°)} = \underline{39.683..} \quad \text{<- eq 1}$$

Answer to part b) in the next equation

$$d = A * \sin(74°) = \underline{24.972..} \quad \text{<- eq 4} \quad || \text{ Answer to B}$$

20.4) a) An angle o has a tangent of T. a) Make use of a right triangle and the Pythagorean theorem to solve for cos(o) and sin(o). b) Make use of the Pythagorean identity to do this problem again. Did you get the same answer?

a) Draw a right triangle with one leg horizontal and one vertical. Label the horizontal leg A. Label the vertical leg B. Let the angle opposite of B be o. Therefore tan(o)=B/A implying that B=A*tan(o) or B=A*T. Re-label side B as B=A*T. Let C be the hypotenuse of the triangle. We have C= sqrt{A^2+(A*T)^2} -> C= sqrt{A^2+A^2*T^2} -> C= sqrt{A^2(1+T^2)} -> C= A*sqrt(1+T^2). Re-label the hypotenuse as C=A*sqrt(1+T^2).

$$\cos(o)= \frac{A}{C} \;\rightarrow\; \cos(o)= \frac{A}{A*\sqrt{1+T^2}} \;\rightarrow\; \cos(o)= \frac{1}{\sqrt{1+T^2}}$$

$$\sin(o)= \frac{B}{C} \;\rightarrow\; \sin(o)= \frac{A*T}{A*\sqrt{1+T^2}} \;\rightarrow\; \sin(o)= \frac{T}{\sqrt{1+T^2}}$$

----

b) sin(o)/cos(o)=tan(o) -> sin(o)/cos(o)=T -> sin(o)=T*cos(o)

therefore

sin^2(o)+cos^2(o)=1 -> {T*cos(o)}^2+cos^2(o)=1 ->

T^2*cos^2(o)+cos^2(o)=1 -> cos^2(o) * (T^2 + 1) = 1 ->

$$\cos^2(o) = \frac{1}{1+T^2} \;\rightarrow$$

$$\cos(o) = \pm \sqrt{\left(\frac{1}{1+T^2}\right)}$$

$$\cos(o) = \pm \frac{1}{\sqrt{1+T^2}}$$

$$\sin(o)=T*\cos(o) \rightarrow \sin(o) = \pm \frac{T}{\sqrt{1+T^2}}$$

27.4) A circle has a radius of R. A chord of this circle has a length of L. What is the area of the smaller region bounded by the chord and the circle?

----

d and b are endpoints of the chord
a is the midpoint of the chord, and the point where a radius of the circle R' intersects the chord
c is the center of the circle

Determining Area of Triangle dcb

angle cab is a right angle .. radius of circle intersecting chords of circle at their midpoints are perpendicular

|ac|^2 + |ab|^2 = |cb|^2 .. Pythagorean theorem ->

|ac|^2 = |cb|^2 - |ab|^2 ->

length of ac = sqrt{R^2 - (L/2)^2}

area triangle dcb =
1/2 * base * height =
1/2 * db * ac =
1/2 * L * ac =
1/2 * L * sqrt{R^2 - (L/2)^2} <-- area triangle dcb

----

angle dcb = 2 * angle acb

proof

```
ac = ac
angle cab = angle cad .. both angles are 90° because radius of
                         circles that intersect chords of circles
                         at their midpoint are perpendicular

da = ba .. a is the midpoint of db
triangle dac is congruent to triangle bac .. SAS
angle acb is congruent to angle acd .. CPCTC

angle acb + angle acd = angle dcb ->
angle acb + angle acb = angle dbc ->
2 * angle acb = angle dbc ->
angle dcb = 2 * angle acb

Proof Complete
----

Determining Area of Sector dcb

Sector dcb is that portion of the circle and its interior that lies
in the interior of angle dcb.  (0°<= angle dcb <= 180°)

area sector dcb =

angle dcb
--------- * area circle =
  360°


2 * angle acb
------------- * Pi * R^2 =
   360deg


          { segment ab }
2 * arcsin{------------}
          {     R      }
------------------------- * Pi * R^2 =
          360°


          {  L  }
          { --- }
          {  2  }
2 * arcsin{------ * Pi * R^2 =
          {  R  }


          {  L  }
2 * arcsin{-----} * Pi * R^2  <-- area sector dcb
          { 2R  }
```

----

area of the smaller region
bounded by the chord and the circle =

area sector dcb - area triangle dcb =

```
          {  L  }              L * sqrt{R^2 - (L/2)^2}
2 * arcsin{-----} * Pi * R^2 - -----------------------
          { 2R  }                         2
```

30.4) What are the orientations of the rays whose end points are at the origin, that are tangent to the parabola y=x^2+1?

If a ray is tangent to a parabola, it intersects the parabola in one point.

y=mx .. line containing the ray
y=x^2+1 .. parabola

These intersect where

mx=x^2+b ->

x^2-mx+b=0 ->

```
   m +/- sqrt(m^2-4)
x= -----------------  <-- the x coordinate where ray and parabola
            2             intersect
```

This line intersects this parabola at one point only if sqrt(m^2-4)=0. Therefore this line intersects the parabola at one point only for m= +/- 2. Therefore the points of tangency are (2,4) and (-2,4).

The slope of the ray (0,0)(2,4) is 4/2 = 2. The slope of the ray (0,0)(-2,4) is 4/-2 = -2.

Possible orientations of the ray with slope 2 are arctan(2)=63.43° and arctan(2)+180° = 243.43°. Given that the point (2,4) is in the 1st quadrant, the orientation of the ray (0,0)(-2,4) is 63.43°.

Possible orientations of the ray with slope -2 are arctan(-2)= -63.43° and arctan(2)+180° = 116.57° Given that the point (-2,4) is in the 2nd quadrant, the orientation of the ray (0,0)(-2,4) is 116.56°.

31.4) From the sunroof of an apartment building, the angle of depression (from the horizontal) to the base of an office building is 51.5° and the angle of elevation (from the horizontal) to the top of the office building is 43.2°. If the office building is 847 ft high, how far apart are the two buildings and how high is the apartment building?

The Picture

Draw the two office buildings as vertical segments side by side whose bottom endpoints rest on a horizontal line. Make the apartment segment shorter than the office segment. Label the top of the apartment segment A. Label the top of the office segment C. Label the bottom of the office segment D. Label a point of the office segment B, such that AB is a horizontal segment. Draw segment AB and mark it as being ? in length. Label segment BD as being ?? in length. Label segment CB as being 847-?? in length. Label angle ABC as being a right angle. Draw segments AC and AD. Label angle CAB as having a measure of 43.2°. Label angle DAB as having a measure of 51.5°.

From the picture we have ...

$$\tan(43.2^\circ) = \frac{847-??}{?} \;\to\; ?*\tan(43.2^\circ)=847-?? \;\to$$

$$??= 847-?*\tan(43.2^\circ)$$

also

$$\tan(51.5^\circ) = \frac{??}{?} \;\to\; ??= ?*\tan(51.5^\circ)$$

so

$$847-?*\tan(43.2^\circ) = ?*\tan(51.5^\circ) \;\to$$

$$?*\tan(51.5^\circ) + ?*\tan(43.2^\circ) = 847 \;\to$$

$$?*\{\tan(51.5^\circ) + \tan(43.2^\circ)\} = 847 \;\to$$

$$? = \frac{847}{\{\tan(51.5^\circ) + \tan(43.2^\circ)\}} = 385.66 \text{ ft} = \text{distance between the two buildings}$$

$$?? = ?*\tan(51.5^\circ) = 484.84 \text{ ft} = \text{height of apt bldg.}$$

32.4) A boat is cruising on a straight course. A rocky point is sighted to the right at an angle of 28 degrees with respect to the direction of travel. (Straight ahead is an angle of zero). The boat continues on for 3.6 miles where the rocky point is spotted again to the right of the boat this time at an angle of 41 degrees with respect to the direction of travel. How close will the boat come to the point?

Draw a vertical segment representing the line of the boats travel. Put a point near the top of this segment, label it A. To the side of A, make a point to represent the location of the rocky point. Label this point R. Below A, mark a point, label it B. B represents the spot where the rocky point was sighted at an angle 41 degrees with respect to the direction of travel. Below B mark another point, call this point C. C represents the point where the rocky point was spotted at an angle of 28 degrees with respect to the direction of travel. Label angle RAB as a right angle. Label ABR as having a measure of 41 degrees. Label angle ACR as having a measure 28 degrees. Label the distance from C to B as 3.6. Label the distance from B to A as ?. Draw the segments AR, BR and CR. Label the distance from A to R as ??.

From the picture we have ..

$$\tan(28°)= \frac{??}{?+3.6} \rightarrow (?+3.6)*\tan(28°)=??$$

also

$$\tan(41°)= \frac{??}{?} \rightarrow ?*\tan(41°)=??$$

so

$$(?+3.6)*\tan(28°)=?*\tan(41°) \rightarrow$$

$$?*\tan(28°) + 3.6*\tan(28°) = ?*\tan(41°) \rightarrow$$

$$?*\tan(41°) - ?*\tan(28°) = 3.6*\tan(28°) \rightarrow$$

$$?*\{\tan(41°) - \tan(28°)\} = 3.6*\tan(28°) \rightarrow$$

$$? = \frac{3.6*\tan(28°)}{\tan(41°) - \tan(28°)} = 5.6702$$

??=?*tan(41°) = 4.9290 miles = <- Solution

34.4) Given the triangle asa=[47°,12,55°], determine all parts of this triangle which are not given, i.e. (sides, angles, and area).

We name the side of length 12, C. We name the side opposite the 47° angle A, and we name the side opposite the 55° angle B. We name the angle opposite side A, a. We name the angle opposite side B, b. We name the angle opposite side C, c. We name the vertex associated with a, a'. We name the vertex associated with b, b'. We name the vertex associated with c, c'. From c' we drop a perpendicular to the line containing C. The point where the perpendicular intersects this line we name d'. Since angles a and b are both less than 90°, d' is on segment (side) C. Mark segment d'b' as being X in length. Mark segment a'd' as being ??=12-X in length. Mark the perpendicular c'd' as being ? in length. This picture and set of facts implies the following two equations.

$$\tan(47°)= \frac{?}{12-X} \quad \text{and} \quad \tan(55°)= \frac{?}{X} \;\to$$

?= (12-X)tan(47°) and ?= X*tan(55°) ->

(12-X)tan(47°) = X*tan(55°) ->

12*tan(47°) - X*tan(47°) = X*tan(55°) ->

X*tan(47°) + X*tan(55°) = 12*tan(47°) ->
X{tan(47°) + tan(55°)} = 12*tan(47°) ->

$$X = \frac{12*\tan(47°)}{\tan(47°) + \tan(55°)} \;\to\; X= 5.1463..$$

tan(55°)=?/X -> ?= X*tan(55°) -> ?= 7.3497..

??=12-X -> ??= 6.8537..

cos(47°)=??/B -> B= ??/cos(47°) -> B= 10.049

cos(55°)=X/A -> A= X/cos(55°) -> A= 8.9723..

angle c = 180° -b -a = 78°

Area Triangle = 1/2 * any side * side's altitude = 1/2 * C * ? = 88.1964

In conclusion

side A= 8.9723	angle a= 47°	area triangle = 44.098
side B= 10.049	angle b= 55°	
side C= 12	angle c= 78°	

35.4) Given the triangle sss=[3,5,7], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

Side of length 3 is A. Side of length 5 is B. Side of length 7 is C. The angle opposite A is a. The angle opposite B is b. The angle opposite C is c. The vertex associated with a is a'. The vertex associated with b is b'. The vertex associated with c is c'. The perpendicular from c to C intersects C at d'. Mark angle b'd'c' as a right angle. Mark the length of b'd' as x. Mark the length of d'a' as ??=7-x. Mark the length of c'd' as ? From this picture we get the following equations.

?^2 = 3^2-x^2 and ?^2 + (7-x)^2 = 5^2

substituting the first equation into the second we get

9-x^2 + (7-x)^2 = 25 ->

9-x^2 + 49 - 14x + x^2 = 25 ->

9 + 49 - 14x = 25 ->

14x = 9 + 49 - 25 ->

14x=33 ->

x=33/14 = 2.3571..

substituting x into ?=sqrt(9-x^2) we get

?= 1.8558..

therefore

tan(b)=?/x -> b=arctan(?/x) or related angle -> b= 38.21°
tan(a)=?/(7-x) -> a=arctan{?/(7-x)} or related angle -> a= 21.79°
c=180°-a-b -> c= 120°

Area triangle = 1/2 * side * side's altitude = 1/2 * C*?= 6.4953

Therefore this triangle has sides 3,5 and 7
The angle opposite side of length 3 is 21.79°
The angle opposite side of length 5 is 38.21°
The angle opposite side of length 7 is 120°
The area of this triangle is 6.4953

36.4) Given the triangle sas=[7,15°,11], determine all parts, of this triangle which are not given, i.e. (sides, angles, and area).

Place the side of the triangle with length 11 horizontally.

Side of length 11 we name C. Side of length 7 we name B. The other other side we name A. The angle opposite A we name a=15°. The angle opposite B is b. The angle opposite C is c. The vertex associated with a is a'. The vertex associated with b is b'. The vertex associated with c is c'.

We drop a perpendicular from c' to C. This perpendicular intersects C at d'. We mark segment a'd' as ?.

cos(15°)=?/B -> ?=B*cos(15°) -> ?= 7*cos(15°) -> ?= 6.7615..

We mark segment c'd' as ??.

sin(15°)=??/B -> ??=B*sin(15°) -> ??=7*sin(15°) -> ??=1.8117..

We mark segment d'b' as ???= 11-?, its length 11-? = 4.2385

tan(b)=??/??? -> b=arctan(??/???) or similar angle = 23.144°..

A=sqrt(??^2+???^2) -> A= 4.6095

c= 180°-a-b -> c= 141.856

area triangle = 1/2 * side * side's altitude = 1/2 * c * ?? = 9.9644..
Therefore

side A= 4.6095	angle a= 15°	area triangle = 9.9644
side B= 7	angle b= 23.144°	
side C= 11	angle c= 141.856°	

3.5.4) Prove: The area of any quadrilateral is equal to the one half of the product of (the lengths of) its diagonals times the sine of the angle o they form. (where o is the acute angle formed by the diagonals or is 90° if d1 and d2 are perpendicular).

The Picture

A,B,C,D are vertices of the quadrilateral. Going from A to B to C to D is going clockwise (or counterclockwise) around the quadrilateral. q is the point there diagonals AC and BD intersect. o is the acute angle formed by diagonals AC and BD.

Area of this quadrilateral ABCD = the area of triangle ABC
+ the area of triangle ADC

Area triangle ABC = 1/2 * base * height = 1/2 * AC * {qB * sin(o)}
Area triangle ADC = 1/2 * base * height = 1/2 * AC * {qD * sin(o)}

Therefore

Area ABCD = 1/2 * AC * {qB * sin(o)} + 1/2 * AC * {qD * sin(o)} ->

Area ABCD = 1/2 * AC * {qB * sin(o) + qD * sin(o)}

Area ABCD = 1/2 * AC * (qB + qD) * sin(o) ->

Area ABCD = 1/2 * AC * BD * sin(o) ->

Area quadrilateral = 1/2 * product of the diagonals *
sine of angle o they form

(where o is the acute angle they form or is 90° if d1 and d2 are perpendicular)

Proof complete

Later we will be able to prove that o can be either (or any of the angles formed by the diagonals.

4.5.4) 'Law of Sines' .. Where A,B are the sides of a triangle, and a,b are the corresponding (opposite) angles,

prove: $$\frac{A-B}{A+B} = \frac{\sin(a)-\sin(b)}{\sin(a)+\sin(b)}$$

It seems likely this theorem is related to the law of sines since its terms are sides of a triangle and the corresponding angles. Having tried and failed to prove this theorem by assuming the law of sines as true (as a postulate), we try a different approach. We assume this theorem is true and use this fact to see if it is possible to prove that the law of sines (or any other identity that we know to be true) is true. If we are successful in doing this we still haven't proved that this identity is true. What we will have succeeded in doing is to prove this identity implies the law of sines (or the other identity which we know to be true), i.e. we have shown that it is possible to prove the law of sines by assuming that this identity is true. If we take this proof and 'run it backwards', if each of the steps in this proof are valid mathematically, then this 'backwards' proof is the proof we seek, i.e. the proof of (A-B)/(A+B)={sin(a)-sin(b)}/{sin(a)+sin(b)}.

Q: If we are able to prove the law of sines (in part) by assuming this theorem is true, does this prove this theorem?

A: No it does not. If C implies D, D does not necessarily imply C.

Assuming this theorem as true, we have

$$\frac{A-B}{A+B} = \frac{\sin(a)-\sin(b)}{\sin(a)+\sin(b)} \rightarrow$$

$$(A-B)\{\sin(a)+\sin(b)\} = (A+B)\{\sin(a)-\sin(b)\} \rightarrow$$

$$A*\sin(a)+A*\sin(b)-B*\sin(a)-B*\sin(b) =$$
$$A*\sin(a)-A*\sin(b)+B*\sin(a)-B*\sin(b) \rightarrow$$

$$A*\sin(b)-B*\sin(a) = -A*\sin(b)+B*\sin(a) \rightarrow$$

$$2A*\sin(b) = 2B*\sin(a) \rightarrow$$

$$\frac{\sin(b)}{B} = \frac{\sin(a)}{A}$$

Now we run this proof backwards, if in doing this each of the steps are mathematically justified, then we will have the proof we desire.

```
sin(b)   sin(a)
------ = ------  ->
  B        A

2A*sin(b) = 2B*sin(a)  ->

A*sin(b)-B*sin(a) = -A*sin(b)+B*sin(a)  ->

A*sin(a)+A*sin(b)-B*sin(a)-B*sin(b) =
A*sin(a)-A*sin(b)+B*sin(a)-B*sin(b)    ->


(A-B){sin(a)+sin(b)} = (A+B){sin(a)-sin(b)}  ->


A-B   sin(a)-sin(b)
--- = -------------
A+B   sin(a)+sin(b)
```

Proof Complete

Several of the steps in this proof are not at all intuitive, which is why I failed at first in trying to prove this identity by using the law of sines as a postulate.

It is left to the student to show that each of the steps in the preceding proof are mathematically justified.

To see the proof of a similar theorem, see proof of 'law of tangents'.

4.6.2) Make use of Classical Geometry to prove the cosine addition formula, i.e. prove $\cos(u+v)=\cos(u)\cos(v)-\sin(u)\sin(v)$. $(0° < u < 90°)$, $(0° < v < 90°)$ and $(0° < u+v < 90°)$

The Picture

C is a unit circle centered at the origin, z. R2 is a ray with end point at z (the origin) and orientation of u+v. R2 lies in the first quadrant. R1 is a ray with endpoint at the origin and orientation of u. R1 also lies in the first quadrant. The orientation of R1 is less than the orientation of R2. (Draw picture with R2 orientation about equal to 60° and R1 orientation about equal to 30°). c is the point where R2 intersects the circle C. b is the point on R1 such that angle zbc is a right angle. a is the point on the positive x axis such that angle zab is a right angle. e is the point on the positive x axis such that angle zec is a right angle. d is the point where the segment ce intersects the segment zb. angle dze has a measure of u. Angle czb has a measure of v. Note that each of the following are right triangles of interest. zed, zab, zbc, dbc, where the middle letter is the vertex of the right angle.

Proof

Justification of each of the following statements is left to the student.

The measure of angle dze we call u, the measure of angle czb we call v. Angle zde is u-90°, therefore angle cdb is also u-90°, therefore the measure of angle dcb is u.

length of segment zc is 1
length of segment zb is 1
length of segment bc is sin(v)
length of segment zb is cos(v)
length of segment ze is cos(u+v)

$$\frac{db}{bc} = \frac{\sin(u)}{\cos(u)} \rightarrow db = \frac{\sin(u)}{\cos(u)}\, bc \rightarrow db = \frac{\sin(u)}{\cos(u)} \sin(v)$$

$$zd = zb - db = \cos(v) - \frac{\sin(u)}{\cos(u)} * \sin(v)$$

$$\frac{ze}{zd} = \frac{\cos(u+v)}{\cos(v) - \dfrac{\sin(u)}{\cos(u)} * \sin(v)} = \cos(u) \rightarrow$$

$$\cos(u+v) = \cos(u)\left\{ \cos(v) - \frac{\sin(v)}{\cos(u)}\sin(u) \right\} \;\rightarrow$$

$\cos(u+v) = \cos(u)\cos(v)-\sin(u)\sin(v)$ <- cosine addition formula
$(0° < u < 90°)$, $(0° < v < 90°)$ and $(0° < u+v < 90°)$

Proof Complete

5.6.2) Make use of Classical Geometry to prove the sine subtraction formula, i.e. prove $\sin(u-v)=\sin(u)\cos(v)-\cos(u)\sin(v)$. $(0° < u < 180°)$, $(0° < v < 90°)$ $(u > v)$

The Picture

C is a unit circle centered at the origin, z. R1 is a ray with end point at the origin z, and an orientation of -v. R2 is a ray with end point at the origin z, and an orientation of u-v. (Draw R1 so so that it has an orientation of about -30°, and draw R2 so that it has an orientation of about 45°). a is the point where the ray R1 intersects the circle C. d is the point where the ray R2 intersects the circle C. c is the point on the x axis such that angle zcd is a right angle. b is the point on the x axis such that angle zba is a right angle. f is the point on R1 such that angle zfd is a right angle. e is the point where df intersects the x axis.

Proof

Justification of each of the following statements is left to the student.

The measure of angle ezf we call v. The measure of angle dzc we call u-v. The measure of dza we call u. The measure of zef is 90°-v. The measure of angle dec is 90°-v. The measure of edc is v.

length of segment zd is 1
length of segment za is 1
length of segment cd is $\sin(u-v)$
length of segment zf is $\cos(u)$
length of segment df is $\sin(u)$

$$\cos(v) = \frac{cd}{de} = \frac{\sin(u-v)}{de} \;\rightarrow\; de = \frac{\sin(u-v)}{\cos(v)}$$

$$\frac{ef}{zf} = \frac{ef}{\cos(u)} = \frac{\sin(v)}{\cos(v)} \rightarrow ef = \frac{\sin(v)}{\cos(v)}\cos(u)$$

$$df = de + ef \rightarrow \sin(u) = \frac{\sin(u-v)}{\cos(v)} + \frac{\sin(v)}{\cos(v)}\cos(u) \rightarrow$$

$$\frac{\sin(u-v)}{\cos(v)} = \sin(u) - \frac{\sin(v)}{\cos(v)}\cos(u) \rightarrow$$

$$\sin(u-v) = \left\{ \sin(u) - \frac{\sin(v)}{\cos(v)}\cos(u) \right\} \cos(v) \rightarrow$$

$\sin(u-v) = \sin(u)\cos(v) - \sin(v)\cos(u)$ $\leftarrow$ sine addition formula
$(0^\circ<u<180^\circ)$, $(0^\circ<v<90^\circ)$ $(u>v)$

Proof Complete

7.7) a) Make use of sin(90°-o)=cos(o) to prove cos(90°-o)=sin(o)
b) Make use of cos(90°-o)=sin(o) to prove sin(90°-o)=cos(o)

a) Proof

substituting 90°-o into o, in sin(90°-o)=cos(o) ->
sin(90°-{90°-o}) = cos(90°-o) ->
sin(90°-90°+o) = cos(90°-o) ->
sin(o) = cos(90°-o) ->
cos(90°-o) = sin(o)

Proof Complete

b) Proof

substituting 90°-o into o, in cos(90°-o)=sin(o) ->
cos(90°-{90°-o}) = sin(90°-o) ->
cos(90°-90°+o) = sin(90°-o) ->
cos(o) = sin(90°-o) ->
sin(90°-o) = cos(o)

Proof Complete

11.7) a) Prove: a) 1+tan^2(o)=sec(o); b) 1+cot^2(o)=csc^2(o)

a) Proof

```
1 + tan^2(o) =

    {sin(o)}2
1 + {------}  =
    {cos(o)}

   sin^2(o)
1+ -------- =
   cos^2(o)

cos^2(o)   sin^2(o)
-------- + ------- =
cos^2(o)   cos^2(o)

cos^2(o)+sin^2(o)
----------------- =
     cos^2(o)

    1
-------- = sec^2(o)   Proof Complete
cos^2(o)
```

18.7) Prove $\frac{\sin(a-b)}{\sin(a+b)} = \frac{\cot(b)-\cot(a)}{\cot(b)+\cot(a)}$

$$\frac{\sin(a-b)}{\sin(a+b)} =$$

$$\frac{\sin(a)\cos(b)-\cos(a)\sin(b)}{\sin(a)\cos(b)+\cos(a)\sin(b)} =$$

$$\frac{\dfrac{\sin(a)\cos(b)-\cos(a)\sin(b)}{\sin(a)\sin(b)}}{\dfrac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\sin(a)\sin(b)}} =$$

$$\frac{\dfrac{\sin(a)\cos(b)}{\sin(a)\sin(b)} - \dfrac{\cos(a)\sin(b)}{\sin(a)\sin(b)}}{\dfrac{\sin(a)\cos(b)}{\sin(a)\sin(b)} + \dfrac{\cos(a)\sin(b)}{\sin(a)\sin(b)}} =$$

$$\frac{\cot(b) - \cot(a)}{\cot(b) + \cot(a)}$$

Proof Complete

20.7) Prove $\frac{1-\tan(o)}{1+\tan(o)} = \frac{\cot(o)-1}{\cot(o)+1}$

Proof

$$\frac{1-\tan(o)}{1+\tan(o)} =$$

```
       1
 1 - ------
     cot(o)
------------- =
       1
 1 + ------
     cot(o)
```

```
cot(o)-1
--------
cot(o)+1
```

Proof Complete

22.7) Prove cos^4(o)-sin^4(o) = 1-2sin^2(o)

Proof

cos^4(o)-sin^4(o) =

{cos^2(o)-sin^2(o)} {cos^2(o)+sin^2(o)} =

{cos^2(o)-sin^2(o)} {1} =

cos^2(o)-sin^2(o) =

cos^2(o)-sin^2(o) + 1 - {sin^2(o)+cos^2(o)} =

1 + cos^2(o)-sin^2(o) - sin^2(o)-cos^2(o) =

1-2sin^2(o)

Proof Complete

```
            tan(a)+tan(b)
24.7) Prove ------------- = cot(a)+cot(b)
             tan(a)tan(b)
```

```
tan(a)+tan(b)
------------- = cot(a)+cot(b) ->
tan(a)tan(b)
```

```
tan(a)+tan(b)     1        1
------------- = ------ + ------ ->
tan(a)tan(b)    tan(a)   tan(b)
```

```
tan(a)+tan(b)    tan(a)+tan(b)
------------- =  -------------
tan(a)tan(b)     tan(a)tan(b)
```

Proof Complete

25.7) Doing this problem will help to prepare you for the problems that follow. .. tan(x)=a: a) cos(x)=?; b) sin(x)=?

```
tan(x)=a  ->
sin(x)/cos(x)=a  ->
sqrt{1-cos^2(x)}/cos(x)=a  ->
sqrt{1-cos^2(x)} = a*cos(x) ->
1-cos^2(x)=(a^2)cos^2(x)  ->
1 = (a^2)cos^2(x)+cos^2(x)  ->
(a^2+1)cos^2(x)= 1
cos^2(x) = 1/(a^2+1)
cos(x) = +/- sqrt{1/(1+a^2)}   <-  The solution to part 'a'. The
                                   solution to part 'b' is similar.
```

11.8) One hose can fill a swimming pool in 4 hours. Another hose can fill a pool in 3 hours. How long will it take for both of these hoses working together to fill the pool? T

A swimming pool -> D= 1 pool

One hose can fill a swimming pool in 4 hours -> $R1 = \dfrac{1\ \text{pool}}{4\ \text{hours}}$

One hose can fill a swimming pool in 3 hours -> $R2 = \dfrac{1\ \text{pool}}{3\ \text{hours}}$

$$\text{Rate total} = R1+R1 = \left(\frac{1}{4} + \frac{1}{3}\right)\frac{\text{pool}}{\text{hour}} = \frac{7}{12}\ \frac{\text{pool}}{\text{hour}}$$

$$D=R*T \rightarrow T= \frac{D}{R}$$

$$T= \frac{D}{R} = \frac{1\ \text{pool}}{\dfrac{7}{12}\ \dfrac{\text{pool}}{\text{hour}}} = \frac{12}{7}\ \text{hour} \quad \text{<---}\quad \text{Answer}$$

5.9) Determine all angles 'o' such that
a) cos(o)=0.21; b) sin(o)=-0.77; c) tan(o)=-1
Determine all angles from -360° to 720° such that
a') cos(o)=0.21; b') sin(o)=-0.77; c') tan(o)=-1

We only calculate solutions for a) and a'), after seeing these
solutions the student should be able to determine solutions for
b) and b') & c) and c').

cos(o)=0.21 ->

one possible value for o is
arccos{cos(o)} = arccos(0.21) = 77.87°..

since cos(o) is an even function we know that an other possible
value for o is -77.87°..

keeping in mind the circle definition of cosine we see that
all possible values of o {where cos(o)=0.21} are

-----

a)

o= 77.87°.. + 360° * n where n = {... -3,-2,-1,0,1,2,3, ...}

also

o= -77.87°.. + 360° * n where n = {... -3,-2,-1,0,1,2,3, ...}

we make use of this result to determine all angles

-----

a')

```
77.87°.. + 360° *  0 =  77.87°..
77.87°.. + 360° *  1 =  437.87°..
77.87°.. + 360° *  2 =  797.87°..  (this result is out of bounds)
77.87°.. + 360° * -1 = -282.12°..
77.87°.. + 360° * -2 = -642.12°.. (This result is out of bounds)

-77.87°.. + 360° *  0 =  -77.87°..
-77.87°.. + 360° *  1 =  282.12°..
-77.87°.. + 360° *  2 =  642.12°..
-77.87°.. + 360° *  3 = 1002.12°..  (this result is out of bounds)
-77.87°.. + 360° * -1 = -437.87°.. (this result is out of bounds)
```

Therefore all angles from -360° to 720° such that cos(o)=0.21 are
-282.12°.. , -77.87°.. , 77.87°.. , 282.12°.. , 437.87°.. ,
642.12°..

7.9) Simplify the following expressions, the simplified forms will not contain any trigonometry functions. Hint: Upon seeing an inverse function, invcos(t) for example, remember to think the words. The angle (returned by the calculator) whose cosine is t. Its best to leave the words (returned by the calculator) out.

a) cos{arccos(x)}; b) cos{arcsin(x)}; c) cos{arctan(x)};

The solutions to the first 3 problems will be given. This should be sufficient to show how to do all of the problems. The student SHOULD do all of the problems.

a) arccos(x) is an angle such that its cosine is x. Therefore
cos{arccos(x)}=x

```
b) cos{arcsin(x)} =    .. making use of cos(x)= +/- sqrt{1-sin^2(x)}
   +/- sqrt[1-sin^2{arcsin(x)}] =
   +/- sqrt{1-x^2}

c) tan{arctan(x)} = x  ->  .. letting o = arctan(x)

   tan(o) = x

   sin(o)
   ------ = x  ->
   cos(o)


   +/- sqrt{1-cos^2(o)}
   -------------------- = x  ->
           cos(o)


   1-cos^2(o)
   ---------- = x^2  ->
    cos^2(o)


   1-cos^2(o) = x^2 * cos^2(o)  ->

   x^2 * cos^2(o) + cos^2(o) = 1  ->

   (x^2 + 1)cos^2(o) = 1  ->

   cos^2(o) = 1/(x^2 + 1)

   cos(o) = +/- sqrt{1/(x^2 + 1)}

                            [  1   ]
   cos{arctan(x)} = +/- sqrt[------]
                            [1+ x^2]
```

12.9) Assuming y=f(x) is a function with an inverse, prove that the function obtained by exchanging the x and the y {i.e. x=f(y)} is the inverse of y=f(x).

Proof

In this proof, f' is the inverse of f.

x=f(y) ->

f'(x)=f'{f(y)} ->

f'(x)=y ->

y=f'(x)

Proof Complete

14.9) Make use of the fact that f{g(x)}=x, implies f is the inverse function of g to calculate the inverse function of each of the following functions. I) These are the same problems as were given in the previous problem. Verify your answers by looking at those answers. a) y=3-2x; b) y=3+x; c) y=1-x.

a) keeping in mind that f'{f(x)}=x, we need to find a function that when 3x-2 is substituted into it, the result is x. We do this by building a function around 3x-2 that converts it to x.

```
(3x-2) + 2                x+2
---------- = x -> f'(x)= ---
    3                      3
```

5.9.2) Determine an inverse of a suitably domain restricted (if required) form of each of the following functions.
a) cos{arcsin(x)};d b) arccos{Sin(x)}; c) sin{arccos(5x-2)};
d) arctan{Sin(ax-b)}

Only the solution to part 'c' will be given here. After you view this solution, you solve this problem using another method.

suitably domain restricted sin{arccos(5x-2)} =
Sin{arccos(5x-2)}

building a function around this function to make this function x, we have

```
cos{arcsin(   Sin{arccos(5x-2)}   )} + 2
----------------------------------------
                   5
```

Therefore the inverse of a suitably domain restricted

$\sin\{\arccos(5x-2)\}$ is

$$\frac{\cos\{\arcsin(x)\} + 2}{5}$$

5.10.4) Graph the following functions over a domain of 2 wavelengths of your choosing. Mark and label all key points. Then make use of a graphing calculator or computer to check your answers.
a) cos(Pi/6 rads-t); b) sin(Pi/6-2t); c) 3cos(60°-2t);
d) y=A*cos(o-bx) (A,b,o are positive).

cos(Pi/6 rads - t) = cos(t - Pi/6 rads)

because cos is an even, therefore to graph cos(Pi/6 -t) you just need to graph cos(t - Pi/6 rads).

-----

sin(Pi/6 rads - 2t) = -sin(2t - Pi/6 rads)

because sin is odd, therefore to graph sin(Pi/6 rads - 2t) you just need to graph -sin(2t - Pi/6 rads).

5.11.1) Make use of $\cos(a)\cos(b)=1/2[\cos(a+b)+\cos(a-b)]$ to derive $\sin(a)\sin(b)=1/2[\cos(a-b)-\cos(a+b)]$.

$\cos(a)\cos(b)=1/2[\cos(a+b)+\cos(a-b)]$ ->

$\cos(90°-a)\cos(90°-b)=$
$1/2[\cos(90°-a+90°-b)+\cos\{90°-a-(90°-b)\}]$ ->

$\sin(a)\sin(b)= 1/2[\cos\{180°-(a+b)\}+\cos(b-a)\}]$ ->

$\sin(a)\sin(b)= 1/2[-\cos\{-(a+b)\}+\cos(b-a)]$ ->

$\sin(a)\sin(b)= 1/2\{\cos(b-a)-\cos(a+b)\}$

8.11.1) The following functions are expressed as two sinusoidals which are added together. Express them as two sinusoidals which are multiplied together. a) $y=\cos(5t)+\cos(9t)$; b) $y=\cos(2t)-\cos(4t)$; c) $y=\cos(at)+\cos(bt)$; d) $y=\sin(at)+\sin(bt)$

This problem isn't as hard as it seems, before looking at the hint below, take a good look at the sinusoidal multiplication identities with this problem in mind and see if you can't figure how to solve this problem for yourself.

Before you look at the solution below, try using the following hint instead.

Hint: Make use of the technique of solving simultaneous linear equations.

We will solve part a, the rest of the problems are similar. To do each of these problems we will be making use of one of the sinusoidal multiplication identities, i.e.

1) $\cos(a)\cos(b)=1/2[\cos(a+b)+\cos(a-b)]$
2) $\sin(a)\sin(b)=1/2[\cos(a-b)-\cos(a+b)]$
3) $\sin(a)\cos(b)=1/2[\sin(a+b)+\sin(a-b)]$

We wish to express $y=\cos(5t)+\cos(9t)$ as two sinusoidals multiplied together. We begin by realizing we will make use of

$\cos(a)\cos(b)=1/2[\cos(a+b)+\cos(a-b)]$

this leads to

$\cos(a)\cos(b)=1/2\{\cos(5t)+\cos(9t)\}$

this implies

```
a+b=5  -> a=5-b  -> 5-b=9+b -> 2b=5-9 -> 2b=-4 -> b=-2
a-b=9  -> a=9+b

a=5-b -> a=5-(-2) -> a=5+2 -> a=7
a=9+b -> a=9+(-2) -> a=9-2 -> a=7
```

Therefore

cos(7t)cos(-2t)=1/2[cos(5t)+cos(7t)] ->

cos(5t)+cos(7t)= 2cos(7t)*cos(2t)

9.11.1) Prove that <u>any</u> sinusoidal multiplied by <u>any</u> other sinusoidal of a different wavelength or frequency can be expressed as two other sinusoidals which are added together.

A*cos(ut-a) and B*cos(vt-b) are two sinusoidals having a different wavelength or frequency which can represent <u>any</u> two sinusoidals that have a different wavelength or frequency, therefore ...

A*cos(ut-a) * B*cos(vt-b)=

(1/2)*A*B[cos{ut-a+vt-b} + cos{ut-a-(vt-b)}]=

(1/2)*A*B[cos{(u+v)t-(a+b)} + cos{(u-v)t-(a-b)}]=

(1/2)*A*B*cos{(u+v)t-(a+b)} + (1/2)*A*B*cos{(u-v)t-(a-b)}

Proof Complete

11.11.1) Prove: <u>Any</u> sinusoidal multiplied by <u>any</u> other sinusoidal of the same wavelength or frequency is a vertically displaced sinusoidal.

A*cos(ut-a) and B*cos(ut-b) are two sinusoidals having the same wavelength or frequency which can represent <u>any</u> two sinusoidals that have the same wavelength or frequency, therefore ...

A*cos(ut-a) * B*cos(ut-b)=

(1/2)*A*B[cos{ut-a+ut-b} + cos{ut-a-(ut-b)}] =

(1/2)*A*B[cos{(u+u)t-(a+b)} + cos{(u-u)t-(a-b)}] =

(1/2)*A*B*cos{(u+u)t-(a+b)} + (1/2)*A*B*cos{(u-u)t-(a-b)} =

(1/2)*A*B*cos{2ut-(a+b)} + (1/2)*A*B*cos{0t-(a-b)}

(1/2)*A*B*cos{2ut-(a+b)} + (1/2)*A*B*cos(a-b)

Proof Complete

Note: Notice that the second term (1/2)*A*B*cos(a-b) contains only constants, {cos(a-b) is a constant also}, therefore the second term itself is a constant or number. therefore the entire expression (1/2)*A*B*cos{2ut-(a+b)} + (1/2)*A*B*cos(a-b) is a vertically displaced sinusoidal.

12.11.1) a) Make use of the cosine-cosine modulation identity

cos(a)cos(b)= 1/2 cos(a+b) + 1/2 cos(a-b)

to derive

cos(ax-m)cos(bx-n)= 1/2 cos{(a+b)x-(m+n)} + 1/2 cos{(a-b)x-(m-n)}

b) Make use of the equation derived in 'a', to prove

(Any sinusoidal of frequency U)*(Any sinusoidal of frequency V)=

{A sinusoidal of frequency (U+V)}+
{A sinusoidal of frequency (U-V)}

derivation

cos(a)cos(b)= 1/2 cos(a+b) + 1/2 cos(a-b)

substituting ax-m into a and bx-n into b and multiplying both sides by A*B we have

A*cos(ax-m) * B*cos(bx-n)=

(A*B)/2 * cos{ax-m+bx-n} + (A*B)/2 * cos{ax-m-(bx-n)}=

(A*B)/2 * cos{(a+b)x-(m+n)} + (A*B)/2 * cos{(a-b)x-(m-n)}

Therefore

A*cos(ax-m) * B*cos(bx-n)=
(A*B)/2 cos{(a+b)x-(m+n)} + (A*B)/2 cos{(a-b)x-(m-n)}

substituting 2Pi*U into a and 2Pi*V into b we have

A*cos{(2Pi*U)x-m} * B*cos{(2Pi*V)x-n}=
A/2 cos{{2Pi*U+2Pi*V}x-(m+n)} + B/2 cos[{2Pi*U-(2Pi*V)}x-(m-n)] ->

A*cos{(2Pi*U)x-m} * B*cos{(2Pi*V)x-n}=
A/2 cos{2Pi(U+V)x-(m+n)} + B/2 cos{2Pi(U-V)x-(m-n)}

Proving that any sinusoidal with frequency U, i.e. A*cos{(2Pi*U)x-m}, multiplied by any other sinusoidal with a frequency of V, i.e. B*cos{(2Pi*V)x-n} equals a sinusoidal with a frequency of U+V, i.e. A/2 cos{2Pi(U+V)x-(m+n)} added to a sinusoidal of frequency U-V, i.e. B/2 cos{2Pi(U-V)x-(m-n)}

Proof Complete

4.11.2) Express each of the following as a product of sinusoidals.
a) cos(10t)+cos(8t); b) sin(5t)+sin(3t); c) cos(2t)+sin(3t);
d) 1+sin(t). e) Simplify the answer you got in part 'd'.

This solution will be for part d only.

We begin by considering the term 1+sin(t) to be the sum of two sinusoidals, where the 1 is a sinusoidal of frequency 0.

we set 1+sin(t)= sin(ut-a)+sin(vt-b)

This implies u=0 and a= -90° and v=1 and b=0

or 1+sin(t)= sin{0t-(-90°)}+sin(1t-0)

Making use of the identity

sin(a)+sin(b)= 2sin{(a+b)/2}cos{(a-b)/2}

and substituting ut-a into a and vt-b into b we have

```
                         {(ut-a)+(vt-b)}   {(ut-a)-(vt-b)}
sin(ut-a)+sin(vt-b)= 2sin{-------------}cos{-------------} ->
                         {      2      }   {      2      }
```

```
                         {ut+vt-a-b}   {ut-vt-a+b}
sin(ut-a)+sin(vt-b)= 2sin{---------}cos{---------} ->
                         {    2    }   {    2    }
```

```
                         {(u+v)t-(a+b)}   {(u-v)t-(a-b)}
sin(ut-a)+sin(vt-b)= 2sin{------------}cos{------------}
                         {     2      }   {      2     }
```

making the aforementioned substitutions

0 -> u; -90° -> a; 1 -> v; 0 -> b

into the equation above we have

sin{0t-(-90°)}+sin(1t-0)=

```
    {(0+1)t-(-90°+0)}   {(0-1)t-(-90°-0)}
2sin{---------------}cos{---------------} ->
    {        2      }   {         2     }
```

```
                     {t+90°}   {-t+90°}
sin(90°)+sin(t)= 2sin{-----}cos{------} ->
                     {  2  }   {  2   }
```

1+sin(t)= 2sin(t/2 + 45°)cos(t/2 - 45°)

5.11.2) Make use of cos(a)+cos(b)=2cos{(a+b)/2}cos{(a-b)/2} to derive sin(a)+sin(b)=2sin{(a+b)/2}cos{(a-b)/2}

cos(a)+cos(b)=2cos{(a+b)/2}cos{(a-b)/2} ->

cos(90°-a)cos(90°-b)=

```
    (90°-a+90°-b)   {90°-a-(90°-b)}
2cos(-----------)cos{-------------} ->
    (     2     )   {       2     }
```

```
                  {180°-(a+b)}   (b-a)
sin(a)sin(b)= 2cos{----------}cos(---) ->
                  {     2    }   ( 2 )
```

sin(a)sin(b)= 2cos{90°-(a+b)/2}cos{(b-a)/2} ->

sin(a)sin(b)= 2sin{(a+b)/2)}cos{(b-a)/2}

8.11.2) Prove sin(a)-sin(b)=2cos{(a+b)/2}sin{(a-b)/2}

We could derive sin(a)-sin(b)= ... using the same method this book used to derive sin(a)+sin(b)= ..., or we could make use of cos(a)+cos(b)= ... to derive it, but the easiest method to do it is to make use of sin(a)+sin(b)= ...

sin(a)+sin(b)= 2sin{(a+b)/2}cos{(a-b)/2} ->

sin(a)+sin(-b)= 2sin{(a-b)/2}cos{(a+b)/2} ->

sin(a)-sin(b)= 2cos{(a+b)/2}sin{(a-b)/2}

Proof Complete

10.11.2) Prove: Any sinusoidal added to any other sinusoidal of the same amplitude, and wavelength or frequency is a single sinusoidal of that same wavelength or frequency.

Proof

Any sinusoidal added to any other sinusoidal of the same amplitude is represented in the following line.

A*cos(ut-a) + B*cos(vt-b) =

A*B*{ cos(ut-a) + cos(vt-b) } =

[making use of identity]
[cos(a)+cos(b)=2cos{(a+b)/2}cos{(a-b)/2} we have]

$$A*B*\ 2\cos\left\{\frac{(ut-a)+(vt-b)}{2}\right\}\cos\left\{\frac{(ut-a)-(vt-b)}{2}\right\}$$

$$A*B*\ 2\cos\left\{\frac{ut+vt-a-b}{2}\right\}\cos\left\{\frac{ut-vt-a+b}{2}\right\}$$

$$2*A*B*\cos\left\{\frac{(u+v)}{2}*t-\frac{(a+b)}{2}\right\}\cos\left\{\frac{(u-v)}{2}*t-\frac{(a-b)}{2}\right\}$$

Proof Complete

11.11.2) cos(a)+sin(b)=
[cos{(a+b)/2}+sin{(a+b)/2}][cos{(a-b)/2}-sin{(a-b)/2}]
Simplify the right side of this equation, i.e. express it so it appears as two sinusoidals multiplied by each other.

```
{   (a+b)    (a+b)} {   (a-b)    (a-b)}
{cos(---)+sin(---)} {cos(---)-sin(---)}=
{   ( 2 )    ( 2 )} {   ( 2 )    ( 2 )}


{   (a+b)    (a+b)}  {   (a-b)    (b-a)}
{cos(---)+sin(---)}  {cos(---)+sin(---)}=
{   ( 2 )    ( 2 )}  {   ( 2 )    ( 2 )}


{   (a+b)    (      a+b)} {   (a-b)    (      b-a)}
{cos(---)+cos(90° - ---)} {cos(---)+cos(90° - ---)}=
{   ( 2 )    (       2 )} {   ( 2 )    (       2 )}


{   (a+b)    (a+b      )}  {   (a-b)    (b-a      )}
{cos(---)+cos(--- - 90°)}  {cos(---)+cos(--- - 90°)}=
{   ( 2 )    ( 2       )}  {   ( 2 )    ( 2       )}


                                 (a+b)   (a-b)
making use of cos(a)+cos(b)= 2cos(---)cos(---) we get
                                 ( 2 )   ( 2 )


    {(a+b)   (a+b      )}       {(a+b)   (a+b      )}
    {(---) + (--- - 90°)}       {(---) - (--- - 90°)}
    {( 2 )   ( 2       )}       {( 2 )   ( 2       )}
2cos{------------------ }  * cos{-------------------} *
    {          2        }       {          2        }


    {(a-b)   (b-a      )}       {(a-b)   (b-a      )}
    {(---) + (--- - 90°)}       {(---) - (--- - 90°)}
    {( 2 )   ( 2       )}       {( 2 )   ( 2       )}
2cos{------------------ }  * cos{------------------ } =
    {          2        }       {          2        }


    (a+b      )                        (a-b      )
2cos(--- - 45°)cos(45°) * 2cos(-45°)cos(--- + 45°)=
    ( 2       )                        ( 2       )


    (a+b      )    sqrt(2)       sqrt(2)     (a-b      )
2cos(--- - 45°) * ------- * 2 * ------- * cos(--- + 45°)=
    ( 2       )       2             2        ( 2       )

    (a+b      )      (a-b      )
2cos(--- - 45°) * cos(--- + 45°)
    ( 2       )      ( 2       )
```

5.11.3) a) Prove the following identity.

. A*cos(wt)+B*sin(wt)=C*cos{wt-c}
. -- C=sqrt(A^2+B^2)
. -- c is the angle such that cos(c)=A/C and sin(c)=B/C

Put an other way, c=invtan(B/A) or c=invtan(B/A)+180 degrees

The following identity makes it quite clear that any two sinusoidals of the same frequency or wavelength added together is a sinusoidal.

A*cos(wt)+B*sin(wt)= C*cos(wt-c) ->

A*cos(wt)+B*sin(wt)= C*{cos(wt)cos(c)+sin(wt)sin(c)} ->

A*cos(wt)+B*sin(wt)= C*cos(c)cos(wt)+C*sin(c)sin(wt) ->

A=C*cos(c) and B=C*sin(c) ->

A=C*cos(c) and B=C*sin(c) -> (these equations derived previously)

A^2 + B^2 = {C*cos(c)}^2 + {C*sin(c)}^2 ->

A^2 + B^2 = (C^2)cos^2(c) + (C^2)sin^2(c) ->

A^2 + B^2 = C^2{cos^2(c)+sin^2(c)} ->

A^2 + B^2 = C^2 * (1) ->

A^2 + B^2 = C^2 ->

C^2 = A^2 + B^2

----

cos(c)= A/C and sin(c)=B/C ->

```
        sin(c)   B/C   B
tan(c)= ------ = --- = - ->
        cos(c)   A/C   A
```

c=arctan(B/A) or c=arctan(B/A)+180°

Proof Complete

```
5.12) a) Transform r^2*tan(o)=1 into (y/x)(x^2+y^2)=1
  b) Transform (y/x)(x^2+y^2)=1 into r^2*tan(o)=1,(step by step).

first we note
                      x                                y
x=r*cos(o) -> cos(o)= -   also   y=r*sin(o) -> sin(o)= -
                      r                                r


                                            y
                                            -
                      sin(o)                r               y
r^2*tan(o)=1 -> r^2 * ------ = 1 ->  r^2 * --- = 1 -> r^2 * - =1 ->
                      cos(o)                x               x
                                            -
                                            r
            y
(x^2+y^2) * - = 1
            x

-----

b) Transform (y/x)(x^2+y^2)=1 into r^2*tan(o)=1,(step by step).

y                 r*sin(o)             sin(o)
- (x^2+y^2)= 1 -> -------- * r^2= 1 -> ------ * r^2 =1 ->
x                 r*cos(o)             cos(o)


tan(o) * r^2 = 1


-------------
6.12  Hint: sin(2o)= 2cos(o)sin(o)
-------------


8.12) a) Prove r=sin(o)/cos^2(o) is a parabola.
  b) Convert the equation you derived in 'a', to polar form.

first we note
                      x                                y
x=r*cos(o) -> cos(o)= -   also   y=r*sin(o) -> sin(o)= -
                      r                                r

r=sin(o)/cos^2(o) ->
```

```
      y
      -
      r       y
r = ----- = ----- ->
     x^2     x^2
     ---     ---
     r^2      r

     r*y           y
r =  --- ->   1 = ---  ->  y= x^2
     x^2          x^2
```

b) Convert the equation you derived in 'a', to polar form.

y=x^2 ->

r*sin(o)= {r*cos(o)}^2 ->

r*sin(o)= r^2*cos^2(o) ->

sin(o)= r*cos^2(o) ->

```
    sin(o)
r= --------
   cos^2(o)
```

11.12) Make use of polar<->Cartesian conversion formulas, to derive the polar distance formula, i.e. derive the fact that the distance between the two points (r2,o2)&(r1,o1) is sqrt{r2^2+r1^2-2*r2*r1*cos(o2-o1)}.

P1 Cartesian form is (x1,y1) : P1 in polar form is (r1,o1)
P2 Cartesian form is (x2,y2) : P2 in polar form is (r2,o2)

distance(P2,P1)^2=

(x2-x1)^2+(y2-y1)=

x2^2-2*x2*x1+x1^2 + y2^2-2*y2*y1+y1^2=

r2^2*cos^2(o2) - 2*r2*cos(o2)*r1*cos(o1) + r1^2*cos^2(o1) +
r2^2*sin^2(o2) - 2*r2*sin(o2)*r1*sin(o1) + r1^2*sin^2(o1)=

r2^2 {cos^2(o2) + sin^2(o2)} + r1^2{cos^2(o1) + sin^2(o1)}-
2*r2*r1*{cos(o2)cos(o1) + sin(o2)sin(o1)}=

r2^2 + r1^2 - 2*r2*r1*cos(o2-o1)

Therefore

distance(P2,P1)= sqrt{r2^2+r1^2-2*r2*r1*cos(o2-o1)}

14.12) a) Determine where the parabola r=cos(o)/sin^2(o) and the line r=1/{cos(o)-sin(o)} intersect. b) Check your answers by calculating where these functions in Cartesian form intersect.

r=cos(o)/sin^2(o) and r=1/{cos(o)-sin(o)} ->

```
 cos(o)          1
-------- = -------------
sin^2(o)   cos(o)-sin(o)
```

cos(o)^2-cos(o)sin(o)=sin^2(o) ->

cos^2(o)-sin^2(o)=cos(o)sin(o) ->

cos^2(o)-sin^2(o)= 1/2 * 2cos(o)sin(o) ->

cos(2o)= 1/2 * sin(2o) ->

```
sin(2o)
------- = 2
cos(2o)
```

tan(2o) = 2, given this we know tan(2o+180°) = 2 ->

2o = arctan(2), 2o + 180° = arctan(2) ->

o=arctan(2)/2 = 31.717° also
o={arctan(2)-180°}/2 = -58.283°

(we probably could solve this problem analytically, however to save work from this point we will use numerical techniques).

substituting each of these angles into r=cos(o)/sin^2(o) we find

we get the points of intersection are

(3.0778,31.717°) and (0.72653,-58.283°)

now we check our answers by substituting these angles into r=1/{cos(o)-sin(o)} we find the points of intersection are

(3.0776,31.717°) and (0.72654,-58.283°)

It remains to check points of intersection by converting the equations to Cartesian form. This part of the answer will not be done here, the student is highly encouraged to do this on their own.

2.13.2) Its given that a(b+c)=ab+ac, prove (b+c)a=ab+ac

(a+b)c =   making use of commutative property
c(a+b) =   making use of distributive property
ca+cb =   making use of commutative property
ac+ab

Proof complete

12.15) The distance from an observer to the closest point p0 on sphere is 17 feet. p1 is an other point on the sphere 17 feet 4 inches away from the observer and angle p1-observer-p0 is 8 degrees. What is the radius of the sphere? [note: The path from the observer to the point P1 does not cross through any portion of the circle].

Use to label the picture

A= the observer
p0= the point on the sphere closest to the observer
p1= another point on the sphere, where the segment p0-p1 doesn't pass through the sphere
o= the angle p0-A-P1 = 8°
d0= the segment A-p1 = 17 feet
d1= the segment A-p1 = 17 feet 4 inches (d1 does not pass through the sphere)
cc= center of the sphere

Trigonometry Solution 1

Applying the law of Cosines we have

r^2=(d0+r)^2+d1^2-2*(d0+r)*d1*cos(o)->

r^2=d0^2+2*d0*r+r^2 + d1^2 - 2d0*d1*cos(o) - 2*r*d1*cos(o) ->

deleting an r^2 from both sides

0=d0^2+2*d0*r + d1^2 - 2d0*d1*cos(o) - 2*r*d1*cos(o)

0=r{2*d0-2*d1*cos(o)}+d0^2+d1^2-2*d0*d1*cos(o) ->

r{2*d1*cos(o)-2*d0}=d0^2+d1^2-2*d0*d1*cos(o)

r={d0^2+d1^2-2*d0*d1*cos(o)}/{2(d1*cos(o)-d0)} <--- answer

where d0 is 17 feet and d1 is 17 feet 4 inches and o is 8°.
the radius of the sphere is r= 17.7546 feet <---- answer

Trigonometry Solution 2 (outline)
for triangle p0-A-p1 we have SAS (d1 o d2) so the triangle is determined. Calculate angle A-P0-P1. Then calculate angle P1-P0-cc. noting that in triangle p0-cc-p1 that segment p0-cc = p1-cc = r. therefore we know that angle p0-p1-cc = angle p1-p0-cc = o2. Therefore for triangle p0-cc-p1 we have ASA therefore the triangle is determined. From here calculate r. -For practice, once you have gone over this solution outline, recreate it without looking (do not memorize).

13.15) a) Prove that the radius of the largest circle that will fit inside a triangle with sides of length A,B and C is r= (2*Area of Triangle)/(A+B+C); [Only a solution for part a is provided here].

c is the centroid of a triangle (ABC) whose sides are A,B and C. [By Classical Geometry, a centroid of a triangle is the point where the angle bisectors of the triangle meet, and is equidistant from each of the three sides]. There are three triangles of note inside triangle ABC. These are triangles cA,cB and cC. Triangle cA is the triangle with side A and c is the vertex opposite side A. Triangle cB is the triangle with side B and c is the vertex opposite side B. Triangle cC is the triangle with side C and c is the vertex opposite side A. Since c is equidistant from all three sides of triangle ABC, there exists a circle whose center is c and is tangent to all three sides of triangle ABC. Let a' be the point where the circle touches A. Let b' be the point where the circle touches B. Let c' be the point where the circle touches C. By classical Geometry: we have ca' is perpendicular to A, cb' is perpendicular to B, and cc' is perpendicular to C. Where r is the radius of the circle, r = ca' = cb' = cc'. Given all this we have ...

area triangle ABC =

area triangle cA + area triangle cB + area triangle cC =

(1/2)*A*r + (1/2)*B*r + (1/2)*C*r =

r * (1/2) * (A+B+C)

Therefore the radius of the largest circle that fits inside a triangle with sides A,B and C is

r= (2*Area of Triangle)/(A+B+C)

15.15) An isosceles triangle has two sides of length 5. The angle opposite the third side is 40°. Without making use of the area of this triangle, determine the radius of the largest circle that will fit inside this triangle.

Draw the picture of the triangle and the circle. As segments, angles and points are named in the word picture below, mark and label and label them. As values for segments or angles given these put these values into the picture you are drawing.

The Picture and Classical Geometry preliminaries.

B and C are the two sides of the triangle of length 5. Label these sides B=5 and C=5. A is the third side of this triangle. a is the angle opposite A. b is the angle opposite B. c is the angle opposite C. a' is the vertex of the triangle associated with a. b' is the vertex of the triangle associated with b. c' is the vertex of the triangle associated with c. z' is the centroid of the triangle, i.e. the point where the angle bisectors of the triangle meet. Cir is the largest circle that fits inside the triangle. (By Classical Geometry) the center of this circle is the centroid of the triangle .. z'. Let x' be the point where Cir intersects B. Let y' be the point where Cir intersects C. Let t' be the point where Cir intersects A. Notice angle c'a't' is common to triangles x'a'z' and c'a't'. Mark this angle with a single angle marker. By classical geometry angles c't'a' and a'x'z' are right angles, mark them as such. Triangles x'a'z' and c'a't' have two angles that are congruent with each other, therefore they are congruent triangles. Given all this, angles x'z'a' and c are congruent angles, mark each of these angles with a double angle marker. Off to the side of the triangle, mark segment t'a' as h. We call the radius of Cir, r. Mark segments x'z' and t'z' as r. By Classical Geometry, angle c'a't' is half of angle c'a'b', therefore angle c'a't' is 20°, mark it as such. c'a'/B = sin(20°) therefore c't'=5*sin(20°). Mark it as such. h/B=cos(20°) therefore h=5*cos(20°), mark it as such.

The fact that triangles a'z'x' and a'c't' are similar leads to the following equation.

```
x'z'   a'z'       r        5cos(20°)-r
---- = ---- -> --------- = ----------- ->
c't'    B      5*sin(20)       5
```

5r=25*sin(20°)*cos(20°)-r*5*sin(20°) ->

5r+5r*sin(20°)=25*sin(20°)*cos(20°) ->

r{1+sin(20°)}=5*sin(20°)*cos(20°) ->

```
   5*sin(20°)*cos(20°)
r= ------------------- ->   r= 1.1974..
        1+sin(20°)
```

17.15) Prove: Every point of the circle, {2*cos(o),2*sin(o)} is
  twice as close to the point (1,0) as to the point (4,0).

On a side note, can you see that the circle {2*cos(o),2*sin(o)} is the circle centered at the origin with a radius of 2? This circle is expressed here in parametric form, with o (theta) being the parameter.

distance squared from circle {2*cos(o),2*sin(o)} to point (1,0) =

{2cos(o)-1}^2 + {2sin(o)}^2 =

4cos^2(o)-2cos(o)+1 + 4sin^2(o) =

4cos^2(o)+4sin^2(o) - 2cos(o) + 1 =

4{cos^2(o)+sin^2(o)} + 1 - 2cos(o) =

4(1) + 1 - 2cos(o) =

5-2cos(o)

distance squared from circle {2*cos(o),2*sin(o)} to point (4,0) =

{2cos(o)-4}^2 + {2sin(o)}^2 =

4cos^2(o)-16cos(o)+16 + 4sin^2(o) =

4cos^2(o)+4sin^2(o) - 16cos(o) + 16 =

4 {cos^2(o)+sin^2(o)} - 16cos(o) + 16 =

4(1) - 16cos(o) + 16 =

20 - 16cos(o)

```
distance^2 from {2*cos(o),2*sin(o)} to (4,0)   20-16cos(o)
-------------------------------------------- = ----------- = 4
distance^2 from {2*cos(o),2*sin(o)} to (4,0)    5-2cos(o)


distance from {2*cos(o),2*sin(o)} to (4,0)
-------------------------------------------- = sqrt(4) = 2
distance from {2*cos(o),2*sin(o)} to (4,0)
```

18.15) a) Determine a parametric expression (see previous problem) of a circle such that each of its points are c times as close to the point (a,0) as to the origin. b) Verify that the points of this circle are c times closer to (a,0) than to the origin.

If following this discussion is difficult, substitute numbers in for a and c. I suggest a=3 and c=2. In any case think of a and c as positive numbers.

a) We start out by trying to find two points on the x axis, both of them c times closer to the point (a,0) as to the point (0,0). One of these points to the left of a, the other to the right. We will assume these are two points of the circle we seek, and due to symmetry considerations, we will further assume these are end points of a diameter of the circle we seek. (due to symmetry, the part of the circle above the x axis should be a mirror image of the part of the circle below the x axis). If these assumptions are correct and we do the rest or our work correctly we will find a circle what meets the given criteria, the student can verify the solution by doing part 'b'.

We first assume a point of the circle is on the left side of a, then we assume it is on the right.

```
For one of these points (?,0), we assume 0<?<a, in this case
(?-0)=c(a-?) ->  ... this equation establishes ? as being c times
                     closer to a than it is to 0.
? = c*a/(1+c)
```

```
For one of these points (??,0), we assume 0<a<??, in this case
c(??-a)= ??-0 ->  ... this equation establishes ?? as being c times
                      closer to a than to 0.
?? = c*a/(c-1)
```

Given that ? and ?? are the x coordinates of a diameter of the circle we seek,

x coordinate of the center of the circle we seek =

(?+??)/2 =

```
c*a/(1+c) + c*a/(c-1)
--------------------- =
          2

c^2*a
-----
c^2-1
                                                  {a*c^2  }
Implying that the center of the circle we seek is {-----,0}
                                                  {c^2-1  }
```

and that the radius of the circle we seek is

(??-?)/2 =

$$\frac{\frac{c*a}{c-1} - \frac{c*a}{1+c}}{2} = \frac{c*a}{c^2-1}$$

Therefore the circle we seek has a parametric expression of

$$\frac{c*a}{c^2-a} * \{\cos(o),\sin(o)\} + \left(\frac{a*c^2}{c^2-1}, 0\right)$$

---

b) This part of the problem is left to the student, the solution is similar to the solution of the previous problem.

20-2.15) Make use of the tangent addition formula to derive a tangent half angle formula which makes use of only tangent function(s) and not cosine functions.

tan(2a)=
tan(a+a)=

$$\frac{2\tan(a)}{1-\tan^2(a)}$$

Therefore $\tan(2a)= \frac{2\tan(a)}{1-\tan^2(a)}$ ->

o=2a, implying o/2=a substututing o into the above equation we have

$$\tan(o)= \frac{2\tan\left(\frac{o}{2}\right)}{1-\tan^2\left(\frac{o}{2}\right)} \;->$$

$$\left\{1-\tan^2\left(\frac{o}{2}\right)\right\}\tan(o) = 2\tan\left(\frac{o}{2}\right) \to$$

$$\tan(o) - \tan^2\left(\frac{o}{2}\right)\tan(o) = 2\tan\left(\frac{o}{2}\right)$$

$$\tan^2\left(\frac{o}{2}\right)\tan(o) + 2\tan\left(\frac{o}{2}\right) - \tan(o) = 0 \to$$ applying the quadratic equation

$$\tan\left(\frac{o}{2}\right) = \frac{-2 \pm \sqrt{4 - 4\tan(o)\{-\tan(o)\}}}{2\tan(o)} \to$$

$$\tan\left(\frac{o}{2}\right) = \frac{-1 \pm \sqrt{1+\tan^2(o)}}{\tan(o)}$$ <-- tangent half angle formula

25.15) Determine an inverse function of a suitably domain restricted form of 3cos(x)+4sin(x).

3cos(x)+4sin(x) = skipping several steps we have

5sin(x+36.87°..) .. this function can be expressed as

y=5sin(x+36.87°..) .. interchanging the x and the y establishes y as the inverse function we seek

x=5sin(y+36.87°..) .. solving now for y in the rest of the steps

x/5 = sin(y+36.87°..) .. introducing a suitable domain restriction we get

x/5 = Sin(y+36.87°..) ->

arcsin(x/5) = arcsin{Sin(y+36.87°..)} ->

arcsin(x/5) = y+36.87°.. ->

y = arcsin(x/5)-36.87°.. <-- The answer we seek

27.15) An arched doorway is in the shape of circular arc centered on top of a rectangle four feet wide and eight feet high. The doorway, is 4 feet wide the circular arc is 4 feet wide. The curricular arc is 5 feet long. What is the area of the doorway?

Note: In solving this problem, radian angle measurement will be used.

Draw a picture and label the picture with the following.

v is the lower left corner of the rectangle
u is the lower right corner of the rectangle
t is the upper right corner of the rectangle
s is the upper left corner of the rectangle
w is the width of the rectangle= u-v= t-s= 4
h is the height of the rectangle= s-v= t-u= 8
a is the arc s-t
La is the length of a = 5
cc is the center of the circle of which a is a part
r is the radius of the circle of which a is a part = cc-t
o is the angle s-cc-t = angle of arc s-t
Aa= area of arch
Ar= area of rectangle= 4x8= 32
Ad= area of doorway= Aa + Ar

......

Area of Arch = Aa=

(Area of the circle, of which a is a part) *
(what proportion of the circle, is a ?)} -
(area of triangle s-cc-t)=

(Pi*r^2) {o/(2pi)} - (1/2 * base * height) =
{Pi*r^2) {o/(2pi)} - [1/2 * r * {r*sin(o)}] =
1/2 * r^2 * {o-sin(o)} ->

1- Area of Arch = Aa = 1/2 * r^2 * {o-sin(o)}

Applying the Law of Cosines we get, w^2=r^2+r^2-2r^2cos(o) ->

2- w^2=2r^2{1-cos(o)}

Length of Arch = La=
{circumference of circle that a is a part of)*
(proportion of circle that a is)=

(2*pi*r)*{o/(2*pi)}= o*r -> La=o*r leading to

3- o=La/r

Substituting equation 3 into equation 2, then converting this equation into a form that can be solved for r using a solve procedure we have, w^2=2r^2{1-cos(La/r)} ->
2r^2{1-cos(la/r)}-w^2=0 -> 2r^2{1-cos(5/r)}-4^2=0 ->

4- 2r^2{1-cos(5/r)}-16=0

Note: Keep in mind, we are using angle radian angle measurement.

r is solved for, by finding the value of r that makes equation 4 equal to 0. A good way to do this is to use a graphing calculator or computer which has the ability to find zeros of functions. In this case an HP48 calculator was used and the answer it returned was ...

5- r=2.210 232 769 08

substituting equation 5 into equation 3 we get
o=la/r = 5/2.210 232 769 08 = 2.262 205 171 30

6- o=2.262 205 171 3 radians or 129.614 808 708 0 degrees

Substituting equations 5 and 6 into equation 1 we get

Area of Arch = Aa = 1/2 * R^2 * {o-sin(o)} = 3.643 956 143 19

Area Doorway = Ad = Aa + Ar = 3.643 956 143 19 + 32 =
35.643 956 143 2

7- Area Doorway = Ad = 35.643 956 143 2 (feet squared) <--- Answer

28.15) Leaning Ladder Problem

Two buildings, A and B stand next to each other forming an alleyway between them. Two ladders, one 3m (meters) long and the other 4m long are in the alleyway. The bottom of the 3m ladder touches the base of building A and leans over against building B. The bottom of the 4m ladder touches the base of building B and leans over onto building A. The point where the ladders cross, is 1m above the ground. What is the width (w) of the alleyway?
b) Express the equation that defines the width of the alley in algebraic form.

1 = distance from where ladders cross to the ground

a = distance from the bottom of 3m ladder to the point on the ground directly below where the ladders cross.

b = distance from the bottom of 4m ladder to the point on the ground directly below where the ladders cross.

$w = a+b$ = width of the alleyway.
u = angle between 3m ladder and ground
v = angle between 4m ladder and ground

$w = 3\cos(u) \rightarrow u=\text{invcos}(w/3)$
$w = 4\cos(v) \rightarrow v=\text{invcos}(w/4)$

$w = a+b = 1/\tan(u) + 1/\tan(v)$
$\quad = 1/\tan\{\text{invcos}(w/3)\} + 1/\tan\{\text{invcos}(w/4)\}$

so w is the appropriate solution of

$$w = 1/\tan\{\arccos(w/3)\} + 1/\tan\{\arccos(w/4)\}$$

We make use of HP48 calculator solve procedure to find the root

$$w = 2.603\ 287\ 754\ 42$$

Solution by
Brian Randall
Electrical Engineer
Hill Air Force Base

we continue, to obtain an answer in algebraic form)

$$w = \frac{1}{\dfrac{\sqrt{(1-w/3)^2}}{(w/3)}} + \frac{1}{\dfrac{\sqrt{1-(w/4)}^2}{(w/4)}}$$

(from here on, each line is implied by the line previous to it)

$$w= \frac{1}{\sqrt{(3/w)^2 - 1}} + \frac{1}{\sqrt{(4/w)^2 - 1}} \rightarrow$$

$$1 = \frac{1}{\sqrt{3^2-w^2}} + \frac{1}{\sqrt{4^2-w^2}}$$ so w = appropriate root of this equation

$$w = 2.603\ 287\ 754\ 42$$

we continue, to obtain an answer in polynomial form)
sqrt(3^2-w^2) * sqrt(4^2-w^2) = sqrt(3^2-w^2) + sqrt(4^2-w^2)

(3^2-w^2)(4^2-w^2) =
(3^2-w^2)+(4^2-w^2) + 2*sqrt(3^2-w^2)*sqrt(4^2-w^2)

(3^2-w^2)*(4^2-w^2)-(3^2-w^2)-(4^2-w^2) =

 2*sqrt(3^2-w^2)*sqrt(4^2-w^2)
{(3^2-w^2)*(4^2-w^2)-(3^2-w^2)-(4^2-w^2)}^2 =
  4*(3^2-w^2)*(4^2-w^2)

(setting u = w^2 and re-arranging)

{(u-3^2)(u-4^2)+(u-3^2)+(u-4^2)}^2 = 4(u-3^2)(u-4^2)

(skipping a few steps)

u^4 - 46U^3 + 763U^2 - 5374u + 13,585 = 0

so u is the appropriate root of this equation

u = 6.777 107 132 31

and w = sqrt(u); w = 2.603 287 754 42 <- answer

30.15) Prove: The radius of a circle circumscribing a triangle with sides A,B,C is (A*B*C)/(4*Area of Triangle)

Hint

Make use of the theorems (see Law of Cosines / Law of Sines section)

The radius r of the circle passing through (circumscribing) all the vertices of any triangle is the length of any one of the sides of the triangle divided by, twice the sin of the angle opposite this side.

and

The area of a triangle is 1/2 of the product of the length of any two sides times the sine of the included angle.

Proof

A triangle has sides A,B,C. The angles opposite these sides are a,b,c respectively. Given this

$$r = \frac{A}{2\sin(a)} = \frac{B}{2\sin(b)} = \frac{C}{2\sin(c)}$$

$$\text{area of Triangle} = \frac{B*C*\sin(a)}{2} = \frac{A*C*\sin(b)}{2} = \frac{A*B*\sin(c)}{2}$$

which leads to

$$r = \frac{B}{2\sin(b)} = \frac{B * \frac{A*C}{4}}{2\sin(b) * \frac{A*C}{4}} = \frac{\frac{A*B*C}{4}}{\frac{2A*C*\sin(b)}{4}} = \frac{\frac{A*B*C}{4}}{\frac{A*C*\sin(b)}{2}} =$$

$$\frac{\frac{A*B*C}{4}}{(\text{area of triangle})} = \frac{A*B*C}{4*(\text{area of triangle})}$$

$$\text{so } r = \frac{A*B*C}{4*(\text{area of triangle})}$$

Proof Complete

```
Preliminary to 31.15)  Prove ..
A*cos(wt)+B*cos(wt)+C*sin(wt)+D*sin(wt)=E*cos(wt)+F*sin(wt) implies
A+B=E and C+D=F

Proof

A*cos(wt)+B*cos(wt)+C*sin(wt)+D*cos(wt)=E*cos(wt)+F*sin(wt)  ->
A*cos(wt)+B*cos(wt)-E*cos(wt)=F*sin(wt)-C*sin(wt)-D*sin(wt)  ->
(A+B-E)cos(wt) = (F-C-D)sin(wt)  ->

A+B-E = 0  and   F-C-D = 0   ->

A+B=E  and   C+D=F

Proof Complete


31.15) Where A<a, B<b, C<c are vectors in polar form prove ..
  a) A*cos(wt-a) + B*cos(wt-b) = C*cos(wt-c) implies
     A<a + B<b = C<c

  b) A<a + B<b = C<c implies
     A*cos(wt-a) + B*cos(wt-b) = C*cos(wt-c)


Proof of 'a'

A*cos(wt-a) + B*cos(wt-b) = C*cos(wt-c)  ->  (using the cosine
                                             (subtraction identity
A*cos(wt)cos(a) + A*sin(wt)sin(a) +
B*cos(wt)cos(b) + B*sin(wt)sin(b) =
C*cos(wt)cos(c) + C*sin(wt)sin(c)      ->

[A*cos(a)]*cos(wt) + [A*sin(a)]*sin(wt) +
[B*cos(b)]*cos(wt) + [B*sin(b)]*sin(wt) =
[C*cos(c)]*cos(wt) + [C*sin(c)]*sin(wt)

and the Preliminary to 34.15 theorem   ->

A*cos(a)+B*cos(b)=C*cos(c)
A*sin(a)+B*sin(b)=C*sin(c)               ->

x coordinate of A<a + x coordinate of B<b = x coordinate of C<c
y coordinate of A<a + y coordinate of B<b = y coordinate of C<c  ->

A<a + B<b = C<c

Proof of 'b'

To prove part 'b', start with the last step of the proof of part 'a'
and from there proceed backwards through each step of proof 'a'.
```

35.15) A and B are sides of a triangle, a and b are the corresponding (opposite) angles. Prove the Law of Tangents, i.e. prove

$$\frac{A-B}{A+B} = \frac{\tan\{(1/2)(a-b)\}}{\tan\{(1/2)(a+b)\}}$$

Proof

For any given triangle,

The Law of Sines

$$\frac{A}{\sin(a)} = \frac{B}{\sin(b)} = \frac{C}{\sin(c)} \rightarrow$$

$$\frac{A}{\sin(a)} = \frac{B}{\sin(b)} = \frac{C}{\sin(c)} = k \rightarrow$$

$A=k*\sin(a)$: $B=k*\sin(b)$: $C=k*\sin(c)$

Therefore

$$\frac{A-B}{A+B} = \frac{k*\sin(a)-k*\sin(b)}{k*\sin(a)+k*\sin(b)} = \frac{k\{\sin(a)-\sin(b)\}}{k\{\sin(a)+\sin(b)\}} = \frac{\sin(a)-\sin(b)}{\sin(a)+\sin(b)} =$$

Making use of the adding (and subtracting) sine functions identities in the 'Sinusoidal Multiplication and Addition' section we have

$$\frac{2\sin\{(1/2)(a-b)\}\cos\{(1/2)(a+b)\}}{2\sin\{(1/2)(a+b)\}\cos\{(1/2)(a-b)\}} =$$

$$\frac{\sin\{(1/2)(a-b)\}}{\cos\{(1/2)(a-b)\}} * \frac{\cos\{(1/2)(a+b)\}}{\sin\{(1/2)(a+b)\}} =$$

$$\tan\{(1/2)(a-b)\} * \frac{1}{\tan\{(1/2)(a+b)\}} = \frac{\tan\{(1/2)(a-b)\}}{\tan\{(1/2)(a+b)\}}$$

Proof Complete

```
38.15) Prove: If a,b,c are angles of a triangle, then
  tan(a)+tan(b)+tan(c) = tan(a)*tan(b)*tan(c)

Proof

tan(a)+tan(b)+tan(c)=

tan(a)+tan(b)+tan{180°-(a+b)}=

tan(a)+tan(b)-tan{a+b-180°)=

tan(a)+tan(b)-tan(a+b)=

               tan(a)+tan(b)
tan(a)+tan(b)- -------------- =
               1-tan(a)tan(b)


                  {          1       }
{tan(a)+tan(t)} * {1 - --------------} =
                  {    1-tan(a)tan(b)}


                  {1-tan(a)tan(b)         1       }
{tan(a)+tan(t)} * {-------------- - --------------} =
                  {1-tan(a)tan(b)   1-tan(a)tan(b)}
                  {  tan(a)tan(b)  }
{tan(a)+tan(t)} * {- --------------} =
                  {  1-tan(a)tan(b)}


                {   tan(a)+tan(b)}
tan(a)*tan(b) * {- --------------} =
                {  1-tan(a)tan(b)}


tan(a)*tan(b) * -tan{(a+b)}=

tan(a)*tan(b) * -tan{a+b - 180°}=

tan(a)*tan(b)*tan{180°-(a+b)}=

tan(a)*tan(b)*tan(c)


Proof complete
```

39.15) Napoleon' s Theorem 1

If the sides of an arbitrary triangle are also sides of 3 different equilateral triangles and these equilateral triangles are pointed outward with respect to the arbitrary triangle. Prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. We refer to this triangle as the Napoleon triangle 1 of the arbitrary triangle.

Note: The centroid of a triangle is where the 3 medians of a triangle meet.

The picture

a,b and c denote the vertex's of the arbitrary triangle and their corresponding angles. A,B and C denote the sides opposite these angles respectively. g is the centroid of the equilateral triangle that includes side C. h is the centroid of the equilateral triangle that includes that includes side A. i is the centroid of the equilateral triangle that includes side B. X is segment ag, Y is segment ai, S is segment gi.

Proof

1: <iac=<gab=30°

applying the law of cosines

2: S^2=X^2+Y^2-2XY*cos(a+60°)

since the centroid of a triangle lies along each median 2/3 of the distance from the vertex to the midpoint of the opposite side, (see Honors Coordinate Geometry by Curtis Blanco for proof)

then

3: Y=(2/3)*{sqrt(3)/2}*B=B/sqrt(3)

4: x=(2/3)*{sqrt(3)/2}*C=C/sqrt(3)

substituting equations 3 and 4 into equation 2 and multiplying both sides by 3 we have

5: 3*S^2=B^2+C^2-2BC*cos(a+60°)

we expand the cosine term, recalling that cos(60°)=1/2 and sin(60°)=sqrt(3)/2. we have

6: cos(a+60°)=cos(a)/2-sin(a)*sqrt(3)/2

substituting equation 6 into equation 5 we have

7: 3S^2=B^2+C^2-BC*cos(a)+sqrt(3)*BC*sin(a)

we apply the law of cosines to triangle abc:

8: A^2=B^2+C^2-2BC*cos(a) -> -2BC*cos(a)= A^2-B^2-C^2

solving for the area of triangle abc

9: area triangle=(1/2)*base*height=(1/2){B}{C*sin(a)}=
(1/2)BC*sin(a)

substituting 1/2 times equation 8 into equation 7 and
substituting 2 times equation 9 into equation 7
and simplifying we have

10: 3S^2=(1/2)(A^2+B^2+C^2)+2*sqrt(3)*(area of triangle)

Since the above equation is symmetrical in A,B, and C, it follows that the triangle (Napoleon's triangle) connecting the three centroids is equilateral. (proof complete)

proof by
Anita Hoskins and Crystal Martin
Dept of Mathematics Education
The University of Georgia

The last part of this proof is admittedly tricky. Lets discuss why this proof is valid. The proof says a,b and c are symmetrical in equation 10. This means if any of these variables exchange places with each other, the right and left sides of equation 10 do not change in value.

If labels, A,B,C and a,b,c of the arbitrary triangle abc were to be rotated clockwise or counter clockwise around triangle abc. The above proof written exactly as it is written now would still apply to this relabeled triangle. However after any rotation of labels, the 'S' of equation 10 would refer to a different side of the "Napoleon triangle 1". Which side depends on if the rotation is clockwise or counterclockwise. This same rotation also exchanges the values within the variables A,B, and C of equation 10 because each of these variables now refer to different sides of triangle abc. However the value of A^2+B^2+C^2 doesn't change neither does the area of triangle abc change, which is why equation 10 is symmetrical with respect to A,B and C.

Since this exact proof can be used to determine any of the sides of the Napoleon triangle 1, and in doing so, would calculate any of the lengths to be the same value, the Napoleon triangle 1 is equilateral. Another discussion of symmetry of variables is at problem 27, in the section Problem set 2 (advanced).